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ON A PROBLEM OF J. L. TAYLOR

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ABSTRACT. Let S be the structure semigroup of a measure algebra $M(G)$ and K be the union of all maximal groups of S. Taylor proposed the following problem: Are there L. C. A. groups G with nontrivial measures concentrated on K? The purpose of this paper is to give a positive solution to this problem.

Let G be a locally compact abelian group with dual group $\hat{G}$. We denote by $M(G)$ the Banach algebra of all bounded regular Borel measures on G under convolution multiplication and total variation norm. In [2], Taylor showed that there is a compact topological semigroup S, called the structure semigroup of $M(G)$, and an order preserving isometry-isomorphism θ of $M(G)$ into $M(S)$ such that:

(1) $\theta(M(G))$ is a weak*-dense L-subalgebra of $M(S)$;

(2) the maximal ideal space of $M(G)$ is identified with $\hat{S}$, the set of all continuous semicharacters on S, and the Gel'fand transform of $\mu \in M(G)$ is given by $\widehat{\mu}(f) = \int_S f\, d\theta\mu$ for $f \in \hat{S}$. $\{\widehat{\mu}(f);\ f \in \hat{S}\}$ is called the spectrum of μ. μ is called symmetric if $\widehat{\mu^*}(f) = \overline{\widehat{\mu}(f)}$ for every $f \in \hat{S}$. Let K be the union of all maximal groups $\{K_p\}_{p \in P}$ of S. Then $K = \{x \in S;\ |f(x)| = 1 \text{ or } 0 \text{ for every } f \in \hat{S}\}$.

Definition. Let $M_K(G)$ be the set of all $\mu \in M(G)$ such that $\theta\mu$ is concentrated on K but $\theta|\mu|(K_p) = 0$ for every $p \in P$.

In [3], Taylor proposes the following problem concerning $M_K(G)$.

Problem. Are there L. C. A. groups G for which $M_K(G) \neq 0$?

The purpose of this paper is to show the existence of a L. C. A. group G such that $M_K(G) \neq 0$. The following is our main

Theorem. *Let $\overline{R}$ be the Bohr compactification of the real line R. Then there exists nonzero $\mu \in M_K(\overline{R})$ so that μ is a positive symmetric measure and the spectrum of μ is a countable set.*

We put $\Lambda_n = \{(\alpha_0, \alpha_1, \ldots, \alpha_n);\ \alpha_0 = 0,\ \alpha_i = 0 \text{ or } 1\ (i = 1, 2, \ldots, n)\}$ $(n = 0, 1, 2, \ldots)$ and $\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n$. For $\alpha \in \Lambda$, we put $|\alpha| = n$ if $\alpha \in \Lambda_n$.

Lemma 1. *There exists a countable family $\{E_\alpha\}_{\alpha \in \Lambda}$ such that E_α is a*

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subset of R $(\alpha \in \Lambda)$ *satisfying the following conditions*:

(1) $E_\alpha \subset E_{\alpha,0}$ *and* $E_\alpha \subset E_{\alpha,1}$ *for* $\alpha \in \Lambda$;

(2) $E_{\alpha,0} \setminus E_\alpha \neq \emptyset$ *and* $E_{\alpha,1} \setminus E_\alpha \neq \emptyset$ *for* $\alpha \in \Lambda$;

(3) *for* α, $\beta \in \Lambda$, $E_\alpha \cap E_\beta = E_{\alpha_0,\alpha_1,\dots,\alpha_j}$ *if* $\alpha_1 = \beta_1, \dots, \alpha_j = \beta_j$ *and* $\alpha_{j+1} \neq \beta_{j+1}$;

(4) $\bigcup_{\alpha \in \Lambda} E_\alpha$ *is an independent set.*

Proof. Since R contains an infinite independent set, it is easy to construct such a family.

We denote by H_α the subgroup of R generated by E_α $(\alpha \in \Lambda)$. The following lemma is clear by Lemma 1.

Lemma 2. *The countable family* $\{H_\alpha\}_{\alpha \in \Lambda}$ *has the following properties*:

(1) $H_\alpha \subset H_{\alpha,0}$ *and* $H_\alpha \subset H_{\alpha,1}$ *for* $\alpha \in \Lambda$;

(2) $H_{\alpha,0}/H_\alpha$ *and* $H_{\alpha,1}/H_\alpha$ *are infinite subgroups for* $\alpha \in \Lambda$;

(3) *for* α, $\beta \in \Lambda$, $H_\alpha \cap H_\beta = H_{\alpha_0,\alpha_1,\dots,\alpha_j}$ *if* $\alpha_1 = \beta_1, \dots, \alpha_j = \beta_j$ *and* $\alpha_{j+1} \neq \beta_{j+1}$.

Let G_α be the annihilator in $\overline{R}$ of H_α $(\alpha \in \Lambda)$. We put $G_\alpha + G_\beta = \{x + y;\ x \in G_\alpha,\ y \in G_\beta\}$; then $G_\alpha + G_\beta$ is a compact subgroup. The following lemma is clear by Lemma 2.

Lemma 3. $\{G_\alpha\}_{\alpha \in \Lambda}$ *is a family of compact subgroups of* $\overline{R}$ *and has the following properties*:

(1) $G_\alpha \supset G_{\alpha,0}$ *and* $G_\alpha \supset G_{\alpha,1}$ *for* $\alpha \in \Lambda$;

(2) $G_\alpha/G_{\alpha,0}$ *and* $G_\alpha/G_{\alpha,1}$ *are compact infinite groups for* $\alpha \in \Lambda$;

(3) *for* α, $\beta \in \Lambda$, $G_\alpha + G_\beta = G_{\alpha_0,\alpha_1,\dots,\alpha_j}$ *if* $\alpha_1 = \beta_1, \dots, \alpha_j = \beta_j$ *and* $\alpha_{j+1} \neq \beta_{j+1}$.

For a compact subgroup $X \subset \overline{R}$, we denote by m_X the normalized Haar measure on X. We can consider $m_X \in M(\overline{R})$. We put $\mu_n = (1/2)^n \sum_{\alpha \in \Lambda_n} m_{G_\alpha}$ $(n = 0, 1, \dots)$. Then $\mu_n \in M(\overline{R})$, $\mu_n \geq 0$ and $\|\mu_n\| = 1$. For $\mu \in M(\overline{R})$, we denote by $\hat{\mu}$ the Fourier-Stieltjes transform of μ. By the definition of $\{\mu_n\}_{n=0}^\infty$ and Lemma 2, we get

Lemma 4. $\{\mu_n\}_{n=0}^\infty$ *has the following properties*:

(1) *If* $\gamma \in H_0$, *then* $\hat{\mu}_n(\gamma) = 1$ *for* $n = 0, 1, 2, \dots$;

(2) *if* $\gamma \in H_{\alpha_0,\alpha_1,\dots,\alpha_{k-1},\alpha_k} \setminus H_{\alpha_0,\alpha_1,\dots,\alpha_{k-1}}$, *then* $\hat{\mu}_n(\gamma) = (1/2)^k$ *for* $n \geq k$ *and* $\mu_n(\gamma) = 0$ *for* $n < k$;

(3) *if* $\gamma \in R \setminus H_\alpha$ *for every* $\alpha \in \Lambda$, *then* $\hat{\mu}_n(\gamma) = 0$, $n = 0, 1, 2, \dots$.

By Lemma 4, $\{\mu_n\}_{n=0}^\infty$ has only one weak*-cluster point μ in $M(\overline{R})$ and has the following properties.

Lemma 5. (1) $\mu \in M(\overline{R})$, $\mu \geq 0$ *and* $\|\mu\| = 1$;

(2) *if* $\gamma \in H_0$, *then* $\hat{\mu}(\gamma) = 1$;

(3) *if* $\gamma \in H_{\alpha_0,\alpha_1,\ldots,\alpha_{k-1},\alpha_k} \setminus H_{\alpha_0,\alpha_1,\ldots,\alpha_{k-1}}$, *then* $\hat{\mu}(\gamma) = (\frac{1}{2})^k$;
(4) *if* $\gamma \in R \setminus H_\alpha$ *for every* $\alpha \in \Lambda$, *then* $\hat{\mu}(\gamma) = 0$.

For $\alpha \in \Lambda$, we put $\Lambda_n^\alpha = \{\beta \in \Lambda_n;\ \alpha_0 = \beta_0, \ldots, \alpha_{|\alpha|} = \beta_{|\alpha|}\}$ for $n \geq |\alpha|$ and $\Lambda^\alpha = \bigcup_{n \geq |\alpha|} \Lambda_n^\alpha$. We put $\mu_n^\alpha = \sum_{\beta \in \Lambda_n^\alpha} (\frac{1}{2})^n m_{G_\beta}$ for $n \geq |\alpha|$. Then $\mu_n^\alpha \geq 0$, $\|\mu_n^\alpha\| = (\frac{1}{2})^{|\alpha|}$ and $\{\mu_n^\alpha\}_{n=|\alpha|}^\infty$ has only one weak*-cluster point μ^α in $M(\overline{R})$, and $\{\mu^\alpha\}_{\alpha \in \Lambda}$ has the following properties.

Lemma 6. (1) $\mu_n = \sum_{\alpha \in \Lambda_n} \mu_n^\alpha$;
(2) *if* $\gamma \in H_\alpha$, *then* $\widehat{\mu^\alpha}(\gamma) = (\frac{1}{2})^{|\alpha|}$;
(3) *for* $\gamma \in H_{\beta_0,\beta_1,\ldots,\beta_{k-1},\beta_k} \setminus H_{\beta_0,\beta_1,\ldots,\beta_{k-1}}$ $(k \geq |\alpha|)$,

$$\begin{aligned} \widehat{\mu^\alpha}(\gamma) &= (\tfrac{1}{2})^k \quad \text{if } (\beta_0, \beta_1, \ldots, \beta_k) \in \Lambda^\alpha, \\ &= 0 \quad\quad\ \ \text{if } (\beta_0, \beta_1, \ldots, \beta_k) \notin \Lambda^\alpha, \end{aligned}$$

(4) *if* $\gamma \in R \setminus H_\alpha$ *for every* $\alpha \in \Lambda$, *then* $\widehat{\mu^\alpha}(\gamma) = 0$.

For a compact subgroup $X \subset \overline{R}$, there exists the strongest L.C.A. group topology on $\overline{R}$ such that X is an open compact subgroup of $\overline{R}$. We denote by $\overline{R}_X$ the resulting L.C.A. topological group. We may consider $M(\overline{R}_X) \subset M(\overline{R})$. For $\lambda_1, \lambda_2 \in M(\overline{R})$, we denote by $\lambda_1 \perp \lambda_2$ if λ_1 is mutually singular with λ_2. For $\lambda \in M(\overline{R})$ and a subset $N \subset M(\overline{R})$, we denote by $\lambda \perp N$ if $\lambda \perp \nu$ for every $\nu \in N$.

Lemma 7. (1) $\mu = \sum_{\alpha \in \Lambda_k} \mu^\alpha$ *for every positive integer* k;
(2) $\mu^\alpha \in M(\overline{R}_{G_\alpha})$, *and* $\mu^\beta \perp M(\overline{R}_{G_\alpha})$ *for* $\beta \neq \alpha$, $|\beta| = |\alpha|$;
(3) *for* $\alpha \neq \beta$,

$$\mu^\alpha * \mu^\beta = (\tfrac{1}{2})^{|\alpha|} (\tfrac{1}{2})^{|\beta|} m_{G_{\alpha_0,\alpha_1,\ldots,\alpha_j}}$$

if $\alpha_1 = \beta_1, \ldots, \alpha_j = \beta_j$ *and* $\alpha_{j+1} \neq \beta_{j+1}$.

Proof. By (3) of Lemmas 2, 5 and 6, we have $\hat{\mu} = \sum_{\alpha \in \Lambda_k} \widehat{\mu^\alpha}$ for every integer k, that is $\mu = \sum_{\alpha \in \Lambda_k} \mu^\alpha$. For $\alpha \neq \beta$ such that $\alpha_1 = \beta_1, \ldots, \alpha_j = \beta_j$ and $\alpha_{j+1} \neq \beta_{j+1}$, we have

$$\begin{aligned} \widehat{\mu^\alpha * \mu^\beta}(\gamma) = \widehat{\mu^\alpha}(\gamma)\widehat{\mu^\beta}(\gamma) &= (\tfrac{1}{2})^{|\alpha|+|\beta|} \quad \text{if } \gamma \in H_{\alpha_0,\ldots,\alpha_j}, \\ &= 0 \quad\quad\quad\quad\ \ \text{if } \gamma \notin H_{\alpha_0,\ldots,\alpha_j}, \end{aligned}$$

by (3) of Lemmas 2 and 6. This shows that

$$\mu^\alpha * \mu^\beta = (\tfrac{1}{2})^{|\alpha|+|\beta|} m_{G_{\alpha_0,\alpha_1,\ldots,\alpha_j}}.$$

Let ϕ be a canonical homomorphism of $\overline{R}$ onto $\overline{R}/G_\alpha$, and for $\lambda \in M(\overline{R})$ we put $\dot{\lambda}(E) = \lambda(\phi^{-1}(E))$ for every Borel set E of $\overline{R}/G_\alpha$. Then $\dot{\lambda} \in M(\overline{R}/G_\alpha)$ and

$\hat{\lambda}(\gamma \circ \phi) = \hat{\lambda}(\gamma)$ for $\gamma \in \widehat{\overline{R}/G_\alpha} = H_\alpha$. If $\gamma \in H_\alpha$ $(\alpha = (\alpha_0, \alpha_1, \dots, \alpha_k))$ then $\gamma \in H_{\alpha_0,\alpha_1,\dots,\alpha_j} \setminus H_{\alpha_0,\alpha_1,\dots,\alpha_{j-1}}$ for some $0 \le j \le k$ and $\hat{\mu}(\gamma) = (½)^j$ by (3) of Lemma 5. Then we have

$$\dot{\mu} = ½ m_{D_{\alpha_0}} + (½)^2 m_{D_{\alpha_0,\alpha_1}} + \dots + (½)^k m_{D_{\alpha_0,\alpha_1,\dots,\alpha_{k-1}}} + (½)^k \delta_0,$$

where $D_{\alpha_0,\alpha_1,\dots,\alpha_j}$ is the annihilator in $\overline{R}/G_\alpha$ of $H_{\alpha_0,\alpha_1,\dots,\alpha_j} \subset H_\alpha$, $m_{D_{\alpha_0,\alpha_1,\dots,\alpha_j}}$ is the normalized Haar measure on $D_{\alpha_0,\alpha_1,\dots,\alpha_j}$ and δ_0 is the point measure at $0 \in \overline{R}/G_\alpha$. Since $H_\alpha/H_{\alpha_0,\alpha_1,\dots,\alpha_j}$ is an infinite group, $m_{D_{\alpha_0,\alpha_1,\dots,\alpha_j}}$ is a continuous measure on $\overline{R}/G_\alpha$. Since μ_n^α $(n \ge |\alpha|)$ is concentrated on G_α, μ^α is concentrated on G_α and $\dot{\mu}^\alpha = (½)^k \delta_0$. Thus $\sum_{\beta \ne \alpha, \beta \in \Lambda_{|\alpha|}} \dot{\mu}^\beta$ is a continuous measure on $\overline{R}/G_\alpha$ and we have $\mu^\alpha \in M(\overline{R}_{G_\alpha})$ and $\mu^\beta \perp M(\overline{R}_{G_\alpha})$ for $\beta \ne \alpha$ and $|\beta| = |\alpha|$.

Remark. By (2) of Lemma 7, $\mu^\alpha \perp \mu^\beta$ if $\alpha \ne \beta$ and $|\alpha| = |\beta|$.

By Lemma 7, we have

Proposition 1. *$\theta\mu(K_p) = 0$ for every maximal group K_p of S.*

Proof. Suppose $\theta\mu(K_p) \ne 0$ for a maximal group K_p of S. Then there is a positive integer n such that $(½)^n < \theta\mu(K_p)$. By (1) of Lemma 7, there is $\alpha \in \Lambda_n$ such that $\theta\mu^\alpha(K_p) \ne 0$. By (2) of Lemma 7, we have $\theta\mu^\beta(K_p) = 0$ for every $\beta \in \Lambda_n (\beta \ne \alpha)$. So we have $\theta\mu(K_p) = \theta\mu^\alpha(K_p) \le \|\mu^\alpha\| = (½)^n$, a contradiction.

For $f \in \hat{S}$ and $f^2 = f$, we put $S_0(f) = \{x \in S;\ f(x) = 0\}$, $S_1(f) = \{x \in S;\ f(x) = 1\}$ and $M(S_j(f)) = \{\mu \in M(\overline{R});\ \theta\mu$ is concentrated on $S_j(f)\}$ $(j = 0, 1)$. Then $M(S_0(f))$ is an L-ideal of $M(\overline{R})$ and $M(S_1(f))$ is an L-subalgebra [2].

Lemma 8. *Let $f \in \hat{S}$ such that $f^2 = f$ and $\tilde{\mu}(f) \ne 0$. Then there exists $\alpha \in \Lambda$ such that:*

(1) $\mu^\alpha \in M(S_1(f))$;

(2) $\mu^\beta \in M(S_0(f))$ *for* $\beta \ne \alpha$ *and* $|\beta| = |\alpha|$.

Proof. Since $\tilde{\mu}(f) \ne 0$, we can decompose $\mu = \lambda_1 + \lambda_2$ $(\lambda_2 \ne 0)$, where $\lambda_1 \in M(S_0(f))$ and $\lambda_2 \in M(S_1(f))$. Suppose that $\mu_n \in M(S_0(f))$ for every integer n. For some integer n_0 such that $(½)^{n_0} < \|\lambda_2\|$, there exists $\chi, \nu \in \Lambda_{n_0}$ such that $\mu^\chi \not\perp \lambda_2$ and $\mu^\nu \not\perp \lambda_2$. Because, $\mu^\chi \perp \mu^{\chi'}$ for $\chi, \chi' \in \Lambda_{n_0}$ and $\chi \ne \chi'$, by the remark of Lemma 7, and $\|\mu^\chi\| = (½)^{n_0}$ for every $\chi \in \Lambda_{n_0}$. By Lemma 7, we have

$$\mu^\chi * \mu^\nu = (½)^{n_0}(½)^{n_0} m_{G_{\chi_0,\chi_1,\dots,\chi_j}},$$

where $\chi_1 = \nu_1, \dots, \chi_j = \nu_j$ and $\chi_{j+1} \ne \nu_{j+1}$, and $\mu^\chi * \mu^\nu \in M(S_0(f))$. Since $\mu^\chi \not\perp \lambda_2$ and $\mu^\nu \not\perp \lambda_2$, we have $\mu^\chi * \mu^\nu \not\perp \lambda_2 * \lambda_2$. Since $\lambda_2 * \lambda_2 \in M(S_1(f))$, we have $\mu^\chi * \mu^\nu \notin M(S_0(f))$, a contradiction. Thus there exists an integer n such that $\mu_n \notin M(S_0(f))$. Let n_1 be the smallest integer such that $\mu_n \notin M(S_0(f))$. Then there exists $\alpha \in \Lambda_{n_1}$ such that $m_{G_\alpha} \in M(S_1(f))$ and $m_{G_\beta} \in$

$M(S_0(f))$ for $\beta \in \Lambda_{n_1}$ and $\beta \neq \alpha$, by (3) of Lemma 3. Since $M(\overline{R}_{G_\alpha}) \subset M(S_1(f))$, we have $\mu^\alpha \in M(S_1(f))$ by (2) of Lemma 7. Suppose that $\mu^\beta \notin M(S_0(f))$ for some $\beta \in \Lambda_{n_1}$ and $\beta \neq \alpha$. Then we have $\mu^\beta * \mu^\alpha \notin M(S_0(f))$. By (3) of Lemma 7, we have

$$\mu^\beta * \mu^\alpha = (\tfrac{1}{2})^{|\beta|+|\alpha|} m_{G_{\alpha_0,\alpha_1,\ldots,\alpha_j}},$$

where $\alpha_1 = \beta_1, \ldots, \alpha_j = \beta_j$, and $\alpha_{j+1} \neq \beta_{j+1}$, and $j < |\alpha| = n_1$. This shows that $\mu^\alpha * \mu^\beta \in M(S_0(f))$, a contradiction. Thus we complete this lemma.

For $f \in \hat{S}$, $f \geq 0$, there exists $g_f \in \hat{S}$, $g_f^2 = g_f$ such that $M(S_1(g_f)) = M(O(f))$, where $O(f) = \{x \in S;\ f(x) = 1\}$ and $M(O(f)) = \{\mu \in M(\overline{R});\ \theta\mu$ is concentrated on $O(f)\}$ [2].

Proposition 2. *$\theta\mu$ is concentrated on K.*

Proof. Let $f \in \hat{S}$ such that $f \geq 0$, $f^2 \neq f$ and $\tilde{\mu}(f) \neq 0$. Let $f = h_f \cdot f$ be the polar decomposition of f, where $h_f^2 = h_f \in \hat{S}$ [2, Lemma 3.3]. Then $\tilde{\mu}(h_f) \neq 0$. By Lemma 8, there exists $\alpha \in \Lambda$ such that $\mu^\alpha \in M(S_1(h_f))$ and $\mu^\beta \in M(S_0(h_f))$ for $\beta \neq \alpha$ and $|\beta| = |\alpha|$. Since $M(\overline{R}_{G_\alpha}) \subset M(S_1(h_f))$ and $m_{G_\alpha} \in M(S_1(g_f))$, we have $M(\overline{R}_{G_\alpha}) \subset M(g_f))$ and $\mu^\alpha \in M(S_1(g_f))$ [3]. Thus we complete the proof of this proposition.

Proposition 3. *μ is a symmetric measure and μ has a countable spectrum.*

Proof. Since $\hat{\mu} \geq 0$, we have $\mu^* = \mu$. Let $f \in \hat{S}$. By the proof of Proposition 2, there exists $\alpha \in \Lambda$ such that $\tilde{\mu}(f) = \widetilde{\mu^\alpha}(f)$. Since $\mu^\alpha \in \widehat{M(\overline{R}_{G_\alpha})}$ and $\tilde{\lambda}(|f|) = \|\lambda\|$ for every positive $\lambda \in M(\overline{R}_{G_\alpha})$, there exists $\gamma \in \overline{R}_{G_\alpha}$ such that $\widetilde{\mu^\alpha}(f) = \widetilde{\hat{\mu}^\alpha}(\gamma)$ [2]. Since $\mu^\alpha \in M(G_\alpha)$, there exists $\eta \in \hat{G}_\alpha \subset R$ such that $\tilde{\mu}(f) = \widetilde{\mu^\alpha}(f) = \hat{\mu}^\alpha(\gamma) = \hat{\mu}^\alpha(\eta)$. By Lemma 6, we have

$$\{\widehat{\mu^\alpha}(\eta);\ \eta \in R\} = \{0, (\tfrac{1}{2})^{|\alpha|}, (\tfrac{1}{2})^{|\alpha|+1}, (\tfrac{1}{2})^{|\alpha|+2}, \ldots\}.$$

Thus we have $\{\tilde{\mu}(f);\ f \in \hat{S}\} = \{0, 1, \tfrac{1}{2}, (\tfrac{1}{2})^2, (\tfrac{1}{2})^3, \ldots\}$. This shows that μ is symmetric and has a countable spectrum.

By Propositions 1, 2 and 3, we have our Theorem.

Corollary. *There is a compact metrizable abelian group G and a nonzero symmetric measure $\mu \in M_K(G)$ so that the spectrum of μ is a countable set.*

Proof. We may assume that E_α is a countable set $(\alpha \in \Lambda)$ in Lemma 1. Let H be the subgroup generated by $\{H_\alpha\}_{\alpha \in \Lambda}$; then H is a countable subgroup of R. Let $H^\perp$ be the annihilator of H in $\overline{R}$; then $H^\perp$ is a compact subgroup of $\overline{R}$. Since $H = \widehat{\overline{R}/H^\perp}$, $\overline{R}/H^\perp$ is a compact metrizable group. Then we can construct $\mu \in M(\overline{R}/H^\perp)$, which has the properties of this corollary, in the same way as in the proof of our Theorem.

REFERENCES

1. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR 27 #2808.

2. J. L. Taylor, *The structure of convolution measure algebras*, Trans. Amer. Math. Soc. 119 (1965), 150–166. MR 32 #2932.

3. ———, *L-subalgebras of* $M(G)$, Trans. Amer. Math. Soc. 135 (1969), 105–113. MR 38 #1472.

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