PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December, 1975

PRIMARY IDEALS IN RINGS OF ANALYTIC FUNCTIONS

NORMAN L. ALLING

ABSTRACT. Primary ideals in the ring of all analytic functions on a noncompact Riemann surface are analyzed with the aid of classical valuation theory.

Let A be the ring of all analytic functions on a noncompact (connected) Riemann surface X. The ideal theory of A has been studied since 1940. (See [1] for references.) The main piece of analysis needed is the general Weierstrass (product) theorem and the Mittag-Leffler theorem. The finitely generated ideals of A are all principal. Each principal maximal ideal M of A is of the form $\{f \in A: f(x) = 0\}$, for some $x \in X$. All other maximal ideals are creatures of the axiom of choice. Maximal and prime ideals of A have been well understood for 20 years. Recently R. Douglas Williams [4] analyzed the primary ideals Q of A. While reviewing Williams' paper, the author found it convenient to cast the definitions and proofs into purely valuation theoretic terms, and found that this seemed to make the ideas of Williams much more transparent and elementary. The purpose of this paper is to elaborate and to expose the results the author found before he wrote his review for *Math Reviews*.

§1 is devoted to the analysis of primary ideals in an abstract valuation ring. It is pure valuation theory that could have been done at any time over the last 40 years. Having failed to find these results in the literature, they have been included here. In §2 results of §1 are applied to the ring A to obtain new proofs of some of Williams' results as well as giving some new results.

1. **Primary ideals in a valuation ring.** Let B be a (proper) valuation ring in a field K, V its valuation, and G its value group. Given a (commutative) ring R, let $\mathfrak{I}(R)$ denote the set of all proper ideals of R, ordered under inclusion, and let $\mathfrak{M}(R)$, $\mathfrak{P}(R)$, and $\mathfrak{Q}(R)$ be the set of all maximal, prime, and primary ideals of R, respectively. A subset S of $G^+ \equiv \{g \in G: g \geq 0\}$ will be called an *upper interval* if $s \in S$ and $t \in G^+$ such that $s \leq t$, implies $t \in S$. Let $\mathfrak{U}(G^+)$ be the set of all proper, upper intervals of G^+.

Received by the editors May 2, 1974 and, in revised form, October 18, 1974.

AMS (MOS) subject classifications (1970). Primary 46E25, 30A98; Secondary 13A15.

Key words and phrases. Primary ideals, analytic functions, valuation theory.

One of the classic facts of valuation theory is that $I \in \mathfrak{I}(B) \mapsto V(I)$ is an injection onto $\mathfrak{U}(G^+)$; thus $\mathfrak{I}(B)$ is totally ordered, and so $\mathfrak{M}(B)$ consists of one point. Let $\sigma(I) \equiv \{g \in G\colon |g| < V(x) \text{ for all } x \in I\}$; thus $\sigma(I)$ is a nonempty interval of G that is symmetric, under subtraction. When such a subset of G is also a subgroup it is called a *convex* (or *isolated*) *subgroup* of G. Another classic result tells us that I is prime if and only if $\sigma(I)$ is a convex subgroup of G. Let $\Delta(G)$ be the set of all convex subgroups of G ordered by anti-inclusion; it is easily seen to be a nonempty totally ordered set that is Dedekind complete. Then $P \in \mathfrak{P}(B) \mapsto \sigma(P)$ is an order preserving injection onto $\Delta(G)$. (See, e.g., [2] as a general reference on classic valuation theory.)

Given $g \in G$, let $W(g)$ be the smallest convex subgroup of G that contains g. Such convex subgroups are called *principal*. W maps G into $\Delta(G)$, and is, in general, not surjective. However im W is dense in $\Delta(G)$. Let $\Sigma(G) \equiv \operatorname{im} W - \{0\}$, and let it be called the *value set* of G. Note that $W(g \pm h) \geq \min(W(g), W(h))$. Given $H \in \Sigma(G)$, let $H^- \equiv \{x \in G\colon W(x) \subsetneqq H\}$; then $H^- \in \Delta(G)$ and H^- is the maximal proper convex subgroup of H. Given $L, T \in \Delta(G)$ such that $L \subset T$, the quotient group T/L can be given a total order making the canonical homomorphism order preserving. $H/H^- \equiv f(H)$ is a nonzero Archimedean (totally ordered) group for each $H \in \Sigma(G)$. Each $f(H)$ admits imbeddings into $\mathbf{R}$. (For more details see [3].)

Proposition 1.1. *Let I be a proper ideal of B; then $I^{1/2}$, its radical, is prime. Further, $I^{1/2}$ is the smallest prime ideal of B that contains I.*

Proof. Well known.

Proposition 1.2. *Let I be a nonprime, proper ideal in B and let $P \equiv I^{1/2}$. There exists a unique $H \in \Sigma(G)$ such that $\sigma(P) = H^-$. Further, given any $x \in P - I$, $H = WV(x)$. Finally, $\sigma^{-1}(H)$ is the maximal prime of B that is contained in I.*

Proof. By (1.1) P is the smallest prime ideal that contains I. Equivalently, $\sigma(P)$ is the largest convex subgroup of $\sigma(I)$. Let $g \equiv V(x)$ and $H \equiv W(g)$; then $g \in (\sigma(I) - \sigma(P)) \cap G^+$. Since $g \in \sigma(I)$, $H^- \subset \sigma(I)$. Since $\sigma(P)$ is the largest convex subgroup of $\sigma(I)$, $H^- \subset \sigma(P)$. Since $x \in P$ $(\equiv I^{1/2})$ there exists $n \in N$ such that $x^n \in I$: i.e., $ng \notin \sigma(I)$; thus $\sigma(I) \subsetneqq H$. So we find that $H^- \subset \sigma(P) \subsetneqq \sigma(I) \subsetneqq H$. Since H^- is the largest proper convex subgroup of H, $H^- = \sigma(P)$. The rest follows easily.

Theorem 1.3. *Let Q be a nonprime, primary ideal in B, and let $P \equiv Q^{1/2}$. By (1.2), $\sigma(P) = H^-$. Let θ be the canonical homomorphism of H onto $f(H)$ $(\equiv H/H^-)$. $\sigma(Q)$ is, of course, a symmetric nonempty proper interval in H.*

Since Q is primary, $\sigma(Q) = \theta^{-1}\theta\sigma(Q)$. Conversely, given a proper ideal I of B which is nonzero, nonprime, and for which $P = I^{1/2}$, then $\sigma(I) = \theta^{-1}\theta\sigma(I)$ implies that I is primary. Thus the proper, nonprime, primary ideals Q of B may be uniquely characterized by giving $H \in \Sigma(G)$ and a nonzero, proper, symmetric interval $\theta\sigma(Q)$ of $f(H)$.

Proof. Let I be a proper, nonprime ideal in B, and let $P \equiv I^{1/2}$; then, by (1.1), P is prime. By (1.2), there exists a unique $H \in \Sigma(G)$ such that $\sigma(P) = H^-$. By (1.1), H^- is the largest convex subgroup of $\sigma(I)$; thus $\sigma(I) \subset H$. Since $I \subsetneqq P$, $(H^- =)\ \sigma(P) \subsetneqq \sigma(I)$. Since I is not prime, $\sigma(I)$ is not a convex subgroup of H: i.e., $\sigma(I) \subsetneqq H$: i.e., $\theta\sigma(I)$ is a nonzero, proper, symmetric interval of $f(H)$. I is primary if and only if given $a, b \in B$, such that $ab \in I$ and $a \notin I$, implies $b \in P$. Let $u, v \in G$; then I is primary if and only if $u + v \notin \sigma(I)$ and $u \in \sigma(I)$ implies $v \notin \sigma(P)$. The contrapositive of this statement is, $v \in \sigma(P)$ implies $u + v \in \sigma(I)$ or $u \notin \sigma(I)$. Since the logical statements "A implies B or not C" and "A and C implies B", are equivalent, we see that I is primary if and only if $v \in \sigma(P)$ and $u \in \sigma(I)$ implies $u + v \in \sigma(I)$, proving the theorem.

Let $F(H)$ be the Dedekind completion of $f(H)$. Then $F(H)$ is either order isomorphic to Z or to $\mathbf{R}$. $\theta(V(Q) \cap H)$ is a nonempty, proper, upper interval of $f(H)^+$. Let $r(Q)$ be g.l.b. $\theta(V(Q) \cap H)$; then $r(Q) > 0$. Let $(-r(Q), r(Q)) \equiv \{x \in f(H): -r(Q) < x < r(Q)\}$ and let $[-r(Q), r(Q)] \equiv \{x \in f(H): -r(Q) \le x \le r(Q)\}$. If $r(Q) \notin f(H)$ these two intervals of $f(H)$ are the same.

2. **Primary ideals in** A. It is easy to see that A_M $(\equiv \{a/b\colon a \in A$ and $b \in A - M\})$ is a (proper) valuation ring of the field of meromorphic functions on X. Let V_M be its valuation and G_M its value group. If M is principal then $G_M \simeq Z$. If not, G_M is isomorphic to a nontrivial ultraproduct of Z taken over a countable set. Assume now that M is nonprincipal. $\Sigma(G_M)$ is a totally ordered set with a maximal element H_0. $\Sigma(G_M) - \{H_0\}$ is an η_1-set. Further, $f(H_0) \simeq Z$, and for each $H \in \Sigma(G_M) - \{H_0\}$, $f(H) \simeq \mathbf{R}$; thus each $f(H)$ is Dedekind complete. (See [1] for details.)

Let Q be a (proper) primary ideal in A; then, of course, $Q^{1/2} \equiv P$ is a prime ideal in A. It is well known that P is contained in a unique maximal ideal M of A. Given an ideal I in A, let $I^e \equiv I \cdot A_M$; then $I \in \mathfrak{I}(A) \mapsto I^e$ maps onto $\mathfrak{I}(A_M) \cap \{A_M\}$. Given $J \in \mathfrak{I}(A_M)$, let $J^c \equiv J \cap A$; then $J \in \mathfrak{I}(A_M) \mapsto J^c$ is a map onto $\mathfrak{I}(A, M) \equiv \{I \in \mathfrak{I}(A)\colon I$ is contained in only one maximal ideal, namely $M\}$. Further $I \in \mathfrak{I}(A, M) \mapsto I^e \in \mathfrak{I}(A_M)$ is an order preserving surjection, and $I^{ec} = I$. If $\mathfrak{P}(A, M) \equiv \mathfrak{P}(A) \cap \mathfrak{I}(A, M)$ and $\mathfrak{Q}(A, M) \equiv \mathfrak{Q}(A) \cap \mathfrak{I}(A, M)$, then $P \in \mathfrak{P}(A, M) \mapsto P^e \in \mathfrak{P}(A_M)$ and $Q \in \mathfrak{Q}(A, M) \mapsto Q^e \in \mathfrak{Q}(A_M)$ are bijections. Let specm A denote the set of all maximal ideals of A. Clearly $\mathfrak{Q}(A)$

is partitioned into $(\mathfrak{Q}(A, M))_{M \in \operatorname{specm} A}$. Thus to analyze $\mathfrak{Q}(A)$ it suffices to analyze each $\mathfrak{Q}(A_M)$; that, of course, is exactly what (1.3) does. (See e.g., [5] for more details about the maps e and c.) Combining all these facts together we get

Theorem 2.1. *$\mathfrak{Q}(A)$ is partitioned into $(\mathfrak{Q}(A, M))_{M \in \operatorname{specm} A}$. $Q \in \mathfrak{Q}(A, M) \to Q^e \in \mathfrak{Q}(A_M)$ is a bijection which carries $\mathfrak{P}(A, M)$ onto $\mathfrak{P}(A_M)$. Let $Q \in \mathfrak{Q}(A, M) - \mathfrak{P}(A, M)$, and let $P \equiv Q^{1/2}$; then $(Q^{1/2})^e = (Q^e)^{1/2}$. There exists a unique $H \in \Sigma(G_M)$ so that $\sigma(P^e) = H^-$. Let θ be the canonical homomorphism of H onto $H/H^- \equiv f(H)$; then $r(Q)$ ($\equiv$ g.l.b. $\theta(V_M(Q^e) \cap H)$) is in $f(H)$. If $H = H_0$ is the smallest proper convex subgroup of G_M, then $f(H_0) \simeq Z$ and so $\theta\sigma(Q^e) = (-r(Q), r(Q))$, $r(Q)$ being > 0. Further, for each choice of $r(Q) > 0$ in $f(H_0)$ there is a unique such Q realizing $r(Q)$ in this way. If $H \in \Sigma(G_M) - \{H_0\}$, $f(H) \simeq \mathbf{R}$, and $\theta\sigma(Q^e)$ is either $(-r(Q), r(Q))$ or $[-r(Q), r(Q)]$, the intervals being different. Further, for each choice of $r(Q) > 0$ in $f(H)$ there exist two such ideals Q which give rise to $(-r(Q), r(Q))$ and $[-r(Q), r(Q)]$. Finally, if M is not principal, $\mathfrak{Q}(A, M) - \mathfrak{P}(A, M)$ has the following order type: $2 \cdot \gamma \cdot \eta_1 + \omega$, using Cantor's conventions, where γ is the order type of $\mathbf{R}$, η_1 the order type of the η_1-set $\Sigma(G_M) - \{H_0\}$, and (as usual), ω is the order type of N.*

Now let us relate this to what Williams did [4]. Given $a \in M - \{0\}$, he defined P_M^a to be $\{f \in A: V_M(f^n) \geq V_M(a)$ for some $n \in N\}$ and $P_{Ma} \equiv \{f \in A: V_M(f) > V_M(a^n)$ for all $n \in N\}$.

Proposition 2.2. *Let $g \equiv V_M(a) \in G_M$ and $H \equiv W(g) \in \Sigma(G_M)$; then $P_{Ma} = \sigma^{-1}(H)$ and $P_M^a = \sigma^{-1}(H^-)$. Thus P_{Ma} and P_M^a are uniquely determined by $WV(a)$ and are prime.*

Next Williams defined $P_M|^a$ to be $\{f \in A: V_M(f) + V_M(b) \geq V_M(a)$, for some $b \in A - P_M^a\}$ and $P_M|_a$ to be $\{f \in A: V_M(f) > V_M(a) + V_M(b)$, for all $b \in A - P_M^a\}$.

Theorem 2.3. *Let $H \equiv WV_M(a)$, θ the canonical homomorphism of H onto $f(H)$, and $r \equiv \theta(V_M(a))$; then $r \in f(H)$. $P_M|^a = \sigma^{-1}\theta^{-1}((-r, r))$ and $P_M|_a = \sigma^{-1}\theta^{-1}([-r, r])$; thus each such primary ideal is uniquely determined by $WV_M(a)$ and $\theta(V_M(a))$. Further, since $f(H)$ is always Dedekind complete, each nonprime, primary ideal in $\mathfrak{Q}(A, M)$ is of this form.*

REFERENCES

1. N. L. Alling, *The valuation theory of meromorphic function fields*, Proc. Sympos. Pure Math., vol. 11, Amer. Math. Soc., Providence, R. I., 1968, pp. 8–29. MR 38 #4700.

2. O. Endler, *Valuation theory*, Springer-Verlag, Berlin and New York, 1972.

3. L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, Oxford; Addison-Wesley, Reading, Mass., 1963, MR 30 #2090.

4. R. D. Williams, *Primary ideals in rings of analytic functions*, Trans. Amer. Math. Soc. 178 (1973), 37–49. MR 47 #9294.

5. O. Zariski and P. Samuel, *Commutative algebra*. Vol. I, University Ser. in Higher Math., Van Nostrand, Princeton, N. J., 1958. MR 19, 833.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627