PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

WIDTH-INTEGRALS OF CONVEX BODIES

ERWIN LUTWAK

ABSTRACT. The width-integrals (Breitenintegrale) as introduced by Blaschke are examined. They are shown to satisfy a cyclic inequality similar to that satisfied by the cross-sectional measures (Quermassintegrale). Other similarities and inequalities between the width-integrals and the cross-sectional measures are shown to exist.

The setting for this paper is Euclidean n-space R^n. Compact convex sets with nonempty interiors are called convex bodies and the space of all convex bodies endowed with the Hausdorff topology is denoted by $\mathcal{K}_n$. We denote the unit n-ball and the unit $(n-1)$-sphere by U and Ω, respectively. For a convex body K, we use $W_i(K)$ to denote its cross-sectional measure (Quermassintegral) of index i. The n-dimensional volume of K is written as $V(K)$. For $\omega \in \Omega$, $b_K(\omega)$ is defined to be half the width of K in the direction ω. Two bodies K and L are said to have similar width if there exists a constant $\lambda > 0$ such that $b_K(\omega) = \lambda\, b_L(\omega)$ for all $\omega \in \Omega$. For $K \in \mathcal{K}_n$ and $p \in \operatorname{int} K$, we use K^p to denote the polar reciprocal of K with respect to the unit sphere centered at p. For reference see Bonnesen and Fenchel [2] and Hadwiger [5].

Width-integrals (Breitenintegrale) were first considered by Blaschke [1, p. 85] and later by Hadwiger [5, p. 266]. The width-integral of index i is defined by:

Definition.

$$B_i(K) = \frac{1}{n}\int_\Omega b_K^{n-i}(\omega)\, dS(\omega) \qquad [i \in R;\ K \in \mathcal{K}_n]$$

where dS is the $(n-1)$-dimensional volume element on Ω.

We note that our definition differs slightly from that of Blaschke in that we multiply by suitable constants to normalize the B_i's. In particular, we have $B_i(U) = \omega_n$ for all $i \in R$ and $B_n(K) = \omega_n$ for all $K \in \mathcal{K}_n$, where ω_n denotes the volume of the unit n-ball.

The width-integral of index i is a map,

$$B_i : \mathcal{K}_n \to R.$$

It is positive, continuous, homogeneous of degree $n-i$ and invariant under

Received by the editors May 2, 1974 and, in revised form, June 28, 1974.

AMS (MOS) subject classifications (1970). Primary 52A40, 52A20.

Key words and phrases. Cross-sectional measures, Hölder's Inequality, Minkowski's Inequality, polar body, support function, width, width-integrals.

motions. In addition, for $i \leq n$ it is also bounded and monotone under set inclusion.

The following simple consequence of Hölder's Inequality [6, p. 140] will be required to prove our main theorem.

Lemma 1. *If f and g are nonzero powers of a (strictly) positive continuous function ϕ defined on Ω and $p > 1$, then*

$$\left[\int_\Omega f(\omega)g(\omega)\,dS(\omega)\right]^p \leq \left[\int_\Omega f^p(\omega)g(\omega)\,dS(\omega)\right]\left[\int_\Omega g(\omega)\,dS(\omega)\right]^{p-1}$$

with equality if and only if ϕ is constant on Ω.

The cross-sectional measures satisfy the cyclic inequality [5, p. 282]:

$$W_i^{k-j}(K)\,W_k^{j-i}(K) \leq W_j^{k-i}(K) \qquad [i < j < k;\ K \in \mathcal{K}_n]$$

with equality if K is an n-ball. The width-integrals satisfy the cyclic inequality given in

Theorem 1.

$$B_j^{k-i}(K) \leq B_i^{k-j}(K)\,B_k^{j-i}(K) \qquad [i < j < k;\ K \in \mathcal{K}_n]$$

with equality if and only if K is of constant width.

To prove this we simply let $f = b_K^{i-j}$, $g = b_K^{n-i}$ and $p = (k-i)/(j-i)$ in Lemma 1.

Lemma 2. $B_{n-1}(K) = W_{n-1}(K) \qquad [K \in \mathcal{K}_n]$.

This is well known [5, pp. 211–212].

Proposition 1.

$$W_i(K) \leq B_i(K) \qquad [i < n-1;\ K \in \mathcal{K}_n]$$

with equality if and only if K is an n-ball.

Proof. In general, we have (see [5, p. 278]):

$$W_i(K) \leq \omega_n^{i+1-n}\,W_{n-1}^{n-i}(K) \tag{1}$$

with equality if and only if K is an n-ball. From Theorem 1 we have

$$\omega_n^{i+1-n}\,B_{n-1}^{n-i}(K) \leq B_i(K) \tag{2}$$

with equality if and only if K is of constant width. The desired result is obtained when we combine (1) and (2) using Lemma 2.

For the cross-sectional measures of the Minkowski sum $K + L$ of two convex bodies K and L we have the following inequality [5, p. 249]:

$$W_i(K + L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)} \qquad [i < n - 1;\ K, L \in \mathcal{K}_n]$$

with equality if and only if K and L are homothetic. For the width-integrals of the Minkowski sum of two convex bodies we have

Theorem 2.

$$B_i(K + L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)} \qquad [i < n - 1;\ K, L \in \mathcal{K}_n]$$

with equality if and only if K *and* L *have similar width.*

To prove this we use Minkowski's Inequality [6, p. 146] in conjunction with the fact that $b_{K+L} = b_K + b_L$.

For the cross-sectional measures of the parallel body $K_\mu = K + \mu U$ we have the Steiner polynomial expression [5, p. 214]:

$$W_j(K_\mu) = \sum_{i=0}^{n-j} \binom{n-j}{i} W_{j+i}(K)\mu^i \qquad [j = 0, 1, \ldots, n;\ \mu > 0].$$

Analogously for the width-integrals of a parallel body we have

Theorem 3.

$$B_j(K_\mu) = \sum_{i=0}^{n-j} \binom{n-j}{i} B_{j+i}(K)\mu^i \qquad [j = 0, 1, \ldots, n;\ \mu > 0].$$

To prove this we merely note that $b_{K_\mu} = b_K + \mu$.

Proposition 2.

$$B_{2n}(K) \leq V(K^p) \qquad [K \in \mathcal{K}_n]$$

with equality if and only if K *is symmetric with respect to* p.

Proof. Let H_K be the support function of K for the origin p. We apply Minkowski's Inequality [6, p. 146] to

$$B_{2n}(K)^{-1/n} = \left[\frac{1}{n}\int_\Omega [2^{-1}H_K(\omega) + 2^{-1}H_K(-\omega)]^{-n}\, dS(\omega)\right]^{-1/n}$$

and obtain

$$B_{2n}(K) \leq \frac{1}{n}\int_\Omega H_K^{-n}(\omega)\, dS(\omega)$$

with equality if and only if $H_K(\omega) = H_K(-\omega)$ for all $\omega \in \Omega$. This is the desired result.

As an immediate corollary to Proposition 2 we have a simple proof of

Corollary. *If* K *is a convex body symmetric with respect to* $c \in \operatorname{int} K$, *then* $\operatorname{Inf}\{V(K^p) \mid p \in \operatorname{int} K\} = V(K^c)$.

We note that this result is a simple consequence of a general (but not easily proven) theorem of H. Guggenheimer [4].

If, for $K \in \mathcal{K}_n$, we let $F(K) = B_{2n}(K) \operatorname{Sup} V^{-1}(K^p)$, where the supremum is taken over all $p \in \operatorname{int} K$, then Proposition 2 (in conjunction with earlier comments) can be used to show that F is a similarity invariant measure of symmetry as defined by B. Grünbaum [3, p. 234].

Proposition 3.

$$B_{n+i}(K) \leq W_{n-i}(K^p) \qquad [0 < i < n;\ K \in \mathcal{K}_n]$$

with equality if and only if K *is an* n*-ball and* p *is its center.*

Proof. In general, we have (see [5, p. 278])

$$\omega_n^{n-i} V^i(K^p) \leq W_{n-i}^n(K^p) \tag{1}$$

with equality if and only if K^p is an n-ball. From Theorem 1 we have

$$B_{n+i}^n(K) \leq \omega_n^{n-i} B_{2n}^i(K) \tag{2}$$

with equality if and only if K is of constant width. The desired result is obtained when we combine (1) and (2) using Proposition 2.

The following theorem was previously obtained by the author [7] as in application of inequalities between dual mixed volumes.

Theorem 4.

$$\omega_n^{n+1-i} \leq W_{n-1}^{n-i}(K) W_i(K^p) \qquad [0 \leq i < n;\ K \in \mathcal{K}_n]$$

with equality if and only if K *is an* n*-ball and* p *is its center.*

Proof. We combine Theorem 1 and Lemma 2 to obtain

$$\omega_n^{n+1-i} \leq W_{n-1}^{n-i}(K) B_{2n-i}(K) \tag{1}$$

with equality if and only if K is of constant width. If $i > 0$, Proposition 3 states

$$B_{2n-i}(K) \leq W_i(K^p) \tag{2}$$

with equality if and only if K is an n-ball and p is its center. If $i = 0$, Proposition 2 states

$$B_{2n-i}(K) \leq W_i(K^p) \tag{2'}$$

with equality if and only if K is symmetric with respect to p. The desired result is obtained when we combine (1) and (2) or (1) and (2′) if $i = 0$.

REFERENCES

1. W. Blaschke, *Vorlesungen über Integralgeometrie*. I, II, Teubner, Leipzig, 1936, 1937; reprint, Chelsea, New York, 1949.

2. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körpern*, Springer, Berlin, 1934; reprint, Chelsea, New York, 1948.

3. B. Grünbaum, *Measures of symmetry for convex sets*, Proc. Sympos. Pure Math., Vol. 7, Amer. Math. Soc., Providence, R. I., 1963, pp. 233–270. MR **27** #6187.

4. H. Guggenheimer, *Polar reciprocal convex bodies*, Israel J. Math. **14** (1973), 309–316. MR **47** #9421.

5. H. Hadwiger, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer, Berlin, 1957.

6. G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, London, 1934.

7. E. Lutwak, *Dual mixed volumes*, Pacific J. Math. **58** (1975), 531–538.

COLUMBIA UNIVERSITY, NEW YORK, NEW YORK