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A FINITE VERSION OF SCHUR'S THEOREM

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ABSTRACT. A necessary and sufficient condition for constancy of curvature in terms of umbilicity of small metric spheres is given.

The following is a remark on a paper of Kowalski in which he has investigated the "properties of hypersurfaces which are characteristic for spaces of constant curvature" and papers of Nomizu [3] and Nomizu and Leung [4]. This type of investigation goes back to Schur [5]. For other results of this type we refer to Cartan [1, Chapter V].

We follow the terminology of Kowalski [2]. By a 'metric sphere' in a Riemann manifold M we understand a subset consisting of points at a fixed distance from a fixed point. If the fixed distance is sufficiently small the metric sphere is a smooth hypersurface. Kowalski [2, Theorem 8] shows that if every sufficiently small metric sphere in M is totally umbilic and $\dim M \geq 4$, then M is conformally euclidean. We shall improve this result as follows:

Theorem. *Let M be a connected C^∞ Riemann manifold of dimension ≥ 3. Then every sufficiently small metric sphere is totally umbilic iff M is of constant curvature.*

Total umbilicity of metric spheres is, roughly speaking, a finite version of the infinitesimal isotropy condition in Schur's well-known theorem: namely if M is a connected Riemann manifold of dimension ≥ 3 such that the sectional curvature depends only on the point (and not on the 2-plane section at the point) then the curvature is actually constant. For this reason we have referred to the above theorem as a finite version of Schur's theorem.

Proof of the theorem. First of all note that a space of constant curvature (without any dimension restrictions) has the stated property, cf. e.g., [2, Proposition 5]. (Alternately note that the property is "visibly" true in the Euclidean case, and since umbilicity is a conformally invariant notion, the property also holds in the spherical and hyperbolic cases.)

Let us now prove the converse. Suppose that every small hypersphere of M is totally umbilic. Fix a point P and a unit tangent vector e_1 at p. Let γ be geodesic through p tangential to e_1. Choose a point q on γ, $q \neq p$,

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sufficiently close to p so that the hypersphere centered at q and passing through p is contained in a normal coordinate neighborhood around q. Now choose a moving frame $\{e_1 \cdots e_m\}$ in a neighborhood of p so that e_1 is tangential to geodesics through q. Let $\{\omega_1, \cdots, \omega_m\}$ be the corresponding dual forms. They satisfy the structure equations

$$(1) \qquad d\omega_a = \sum_{b=1}^{m} \omega_{ab}\omega_b, \quad d\omega_{ab} = \sum_{c=1}^{m} \omega_{ac}\omega_{cb} + \Omega_{ab}, \quad \omega_{ab} + \omega_{ba} = 0.$$

We have $\omega_1 = dr$ where r is the radial parameter measured from q. Hence $0 = d\omega_1 = \Sigma_{a=2}^{m} \omega_{1a}\omega_a$. It follows that ω_{1a} is a linear combination of $\omega_2, \cdots, \omega_m$ only. On the other hand, ω_{1a} define the second fundamental forms of metric spheres centered at q; hence by the total umbilicity of the metric spheres they must be of the form $\omega_{1a} = \lambda\omega_a + \mu_a\omega_1$, where λ, μ_a are smooth functions. Combining these two observations we see that $\mu_a = 0$ and so

$$\omega_{1a} = \lambda\omega_a.$$

Taking the exterior derivative and using the structure equations, we have

$$\begin{aligned} d\omega_{1a} &= d\lambda\omega_a + \lambda\omega_{a1}\omega_1 + \sum_{b\geq 2} \lambda\omega_{ab}\omega_b \\ &= \sum_{b\geq 2} \omega_{1b}\omega_{ba} + \Omega_{1a} = \lambda \sum_{b\geq 2} \omega_{ab}\omega_b + \Omega_{1a}. \end{aligned}$$

Consequently

$$\Omega_{1a} = d\lambda\omega_a + \lambda\omega_1\omega_{1a} = (\partial\lambda/\partial r + \lambda^2)\omega_1\omega_a + \text{other terms}.$$

This shows that at p the sectional curvature of the 2-plane spanned by e_1 and e_a is $-(\partial\lambda/\partial r + \lambda^2)$. Since this expression is independent of a, it follows that the sectional curvature of the 2-plane containing e_1 is the same. Since e_1 is arbitrary, we see that all sectional curvatures at p are the same. If $\dim M \geq 3$, by the Schur theorem it follows that M is of constant curvature. A further use of Codazzi's equation implies that λ is in fact constant on a metric sphere. Q.E.D.

Remark. The above theorem, just as the Schur's theorem, of course, breaks down when $\dim M = 2$; in fact any curve on a 2-dimensional manifold is trivially totally umbilic. In this case we can restore the theorem by making a stronger hypothesis.

Theorem. *Let M be a 2-dimensional connected Riemann manifold such that every sufficiently small metric sphere is of constant geodesic curvature; then M is of constant curvature.*

(In Kowalski's terminology constancy of geodesic curvature of a metric sphere is the same as a metric sphere being a U-sphere. So this theorem is a partial improvement of [2, Theorem 3]. Converse of course is true and trivial.)

Proof. In the above notation, constancy of geodesic curvature means that the function λ is constant on metric spheres—hence it is a function of r. So the sectional curvature $-(d\lambda/dr + \lambda^2)$ is constant (say c) on a metric sphere (say s). Now considering the metric spheres with centers on S, and continuing this way, we see that the set of points where the sectional curvature is c is open and closed. Since M is connected, it must be of constant curvature. Q.E.D.

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