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DEGREE SEQUENCES IN COMPLEXES AND HYPERGRAPHS

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ABSTRACT. Given an n-complex K and a vertex v in K, the *n-degree* of v is the number of n-simplexes in K containing v. The set of all n-degrees in a complex K is called its *n-degree sequence* when arranged in nonincreasing order. The question "Which sequences of integers are n-degree sequences?" is answered in this paper. This is done by generalizing the iterative characterization for the 1-dimensional (graphical) case due to V. Havel. A corollary to this general theorem yields the analogous generalization for k-graphs. The characterization of P. Erdős and T. Gallai is discussed briefly.

0. **Introduction.** Given a complex or hypergraph, the notion of degree of a vertex generalizes in an obvious way. Arranging these degrees in nonincreasing order, one obtains a degree sequence for the complex or hypergraph, and one may ask which finite sequences of numbers are degree sequences. For graphs this question has been decided by the iterative characterization due to V. Havel [4] and by the characterization due to P. Erdős and T. Gallai [2]. The first of these results is generalized to complexes, and, through a corollary, to hypergraphs. The second of these is mentioned briefly.

1. **Definitions.** A *hypergraph* is a collection H of subsets of some finite set V. The sets in H are called *hyperedges* or simply *edges*, while the elements of V are called *vertices*. This definition differs slightly from that given in Berge [1, p. 373], where the edges are required each to contain at least one vertex and where it is required that $\bigcup_{h \in H} h = V(H)$. A hypergraph in which every edge has cardinality k is called a *k-graph*. If K is a *hereditary* hypergraph, i.e. every subset of an edge in K is an edge in K, then K is called a *complex* and the members of K are called *simplexes*. This definition is equivalent to the definition of a "finite abstract simplicial complex", as given in [5, p. 41]. In a complex the *dimension of a simplex* x is the number $|x| - 1 = m$, whence x is called an *m-simplex*. The *dimension of a complex* K is the maximum dimension n of any of its simplexes, whence K is called an *n-complex*.

In a hypergraph H the *degree of a vertex* v, written $d(v)$, is the cardinality of the set $\{h \in H: v \in h\}$. In a complex K the *n-degree of a vertex*

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v, written $d^n(v)$, is the cardinality of the set $\{x \in K\colon v \in x$ and x is an n-simplex$\}$.

Let K be a complex and v a vertex not in K. Denote by $v * K$ the complex $\{x \cup \{v\}\colon x \in K \cup \{\emptyset\}\}$. If $u \in V(K)$, denote by $lk(u, K)$ the complex $\{x - \{v\}\colon v \in x \in K\}$. These new complexes are called, respectively, the *cone at* v *over* K and the *link of* u *in* K.

The decreasing n-degree (*degree*) *sequence* of a complex K (hypergraph H), is the sequence obtained by arranging the set $\{d^n(v)\colon v \in V(K)\}$ ($\{d(v)\colon v \in V(H)\}$) in nonincreasing order. A finite sequence of nonnegative integers is called a $(0, n)$-*sequence* (*k-graphical sequence*) when, arranged in nonincreasing order, it is the decreasing n-degree (degree) sequence of some n-complex (k-graph).

2. **A generalization of Havel's theorem.** Obviously, both notions of complex and hypergraph generalize the concept of a graph. Havel's theorem[1] may accordingly be stated as a characterization either of $(0, 1)$-sequences or of 2-graphical sequences. Since the results to follow are obtained first for complexes, we choose the former presentation:

Theorem 1 (Havel). *The nonincreasing sequence* $d_1, d_2, \ldots, d_p$ *of nonnegative integers is a* $(0, 1)$-*sequence if and only if the modified sequence*

$$\text{(i)} \qquad d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$$

is a $(0, 1)$-*sequence.*

Given a $(0, 1)$-sequence like (i) above, one obtains a 1-complex corresponding to the sequence $d_1, d_2, \ldots, d_p$ in the following manner: Let K' be a 1-complex whose 1-degrees are given by (i) and let $v_2, v_3, \ldots, v_{d_1+1}, v_{d_1+2}, \ldots, v_p$ be the corresponding vertices of K'. Adjoin an additional vertex v_1 to K' along with the 1-simplexes $\{v_1, v_2\}, \{v_1, v_3\}, \ldots, \{v_1, v_{d_i+1}\}$ and observe in the resulting 1-complex K that $d^n(v_i) = d_i$, $i = 1, 2, \ldots, p$.

Given a $(0, 1)$-sequence like $d_1, d_2, \ldots, d_p$ above, and a corresponding 1-complex K, one obtains a complex K' corresponding to (i) only after modifying the 1-simplexes of K in a certain manner. Havel's method for doing this generalizes in a rather direct fashion.

Theorem 2. *The nonincreasing sequence* $d_1, d_2, \ldots, d_p$ *is a* $(0, n)$-*sequence if and only if there is a decreasing* $(0, n-1)$-*sequence* $d'_1, d'_2, \ldots, d'_q$ *for which the following conditions hold*

(i) $\sum_{i=1}^{q} d'_i = nd_1$.

(ii) $d_2 - d'_1, d_3 - d'_2, \ldots, d_{q+1} - d'_q, d_{q+2}, \ldots, d_p$ *is a* $(0, n)$-*sequence.*

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1. Also discovered by S. Hakimi [3] seven years later.

Proof. In what follows, we will use the symbols K, K' and K'' for various complexes. The n-degrees (or $(n-1)$-degrees) of vertices in these complexes will be denoted by d, d' and d'' respectively. The sufficiency of the above conditions is shown first.

Let K'' be an n-complex with vertices $v_2, v_3, \ldots, v_p$ such that $d''^n(v_i) = d_i - d'_{i-1}$, $i = 2, 3, \ldots, q+1$, while $d''^n(v_i) = d_i$, $i = q+2, \ldots, p$. Let K' be an $(n-1)$-complex with vertices $v_2, v_3, \ldots, v_{q+1}$ such that $d'^{n-1}(v_i) = d'_{i-1}$, $i = 2, 3, \ldots, q+1$. Then, by condition (i), $\sum_{i=2}^{q+1} d'^{n-1}(v_1) = nd_i$. Let $K = K'' \cup v_1 * K'$, where v_1 is some vertex not in K''. In K we find that $d^n(v_1)$ is just the number of $(n-1)$-simplexes in K', which can also be written as $(1/n)\sum_{i=2}^{q+1} d'^{n-1}(v_i) = d_1$. For each $i = 2, 3, \ldots, q+1$, we have

$$d^n(v_i) = d''^n(v_i) + d'^{n-1}(v_i) = d_i - d'_{i-1} + d'_{i-1} = d_i,$$

while for each $i = q+2, \ldots, p$ we have $d^n(v_i) = d''^n(v_i) = d_i$. Therefore, $d_1, d_2, \ldots, d_p$ is a $(0, n)$-sequence.

The necessity of conditions (i) and (ii) is now shown: Let K be an n-complex whose decreasing n-degree sequence is $d_1, d_2, \ldots, d_p$ and whose vertices are $v_1, v_2, \ldots, v_p$ with $d^n(v_i) = d_i$, $i = 1, 2, \ldots, p$.

Let K' be the union of $lk(v_1, K)$ with the set of 0-simplexes of K and let $K'' = K - v_1$. Observe for any vertex v_i, $i = 2, 3, \ldots, p$, that $d^n(v_i) = d'^{n-1}(v_i) + d''^n(v_i)$. Consider now the sequence $v_2, v_3, \ldots, v_p$ and define the *slope* of this sequence in K' to be the sum $\sum_{i=2}^{p} i \cdot d'^{n-1}(v_i)$. This sum measures the extent to which the sequence $d'^{n-1}(v_2), d'^{n-1}(v_3), \ldots, d'^{n-1}(v_p)$ tends to be increasing. Two cases are now distinguished:

Case 1. The numbers $d'^{n-1}(v_2), d'^{n-1}(v_3), \ldots, d'^{n-1}(v_p)$ form a nonincreasing sequence: let q be the largest integer satisfying $d'^{n-1}(v_{q+1}) > 0$. It is observed that

$$\sum_{i=2}^{q+1} d'^{n-1}(v_i) = n \cdot \alpha_{n-1}(K') = n \cdot d^n(v_1),$$

where $\alpha_{n-1}(K')$ denotes the number of $(n-1)$-simplexes in K'. Clearly, condition (i) is satisfied by the degree sequence of K'. The sequence $d'_1, d'_2, \ldots, d'_p$ defined by $d'_i = d'^{n-1}(v_{i+1})$, $i = 1, 2, \ldots, q$, is a decreasing $(0, n-1)$-sequence, and the n-complex K'' has the degree sequence $d_2 - d'_1, \ldots, d_{q+1} - d'_q, d_{q+2}, \ldots$, where $d_i = d^n(v_i)$, $i = 2, 3, \ldots, p$, satisfying condition (ii). This establishes the necessity of the two conditions in Case 1.

Case 2. The numbers $d'^{n-1}(v_2), d'^{n-1}(v_3), \ldots, d'^{n-1}(v_p)$ do not form a nonincreasing sequence. Let k be any integer for which $d'^{n-1}(v_k) < d'^{n-1}(v_{k+1})$. Since $d'^{n-1}(v_k) < d'^{n-1}(v_{k+1})$, there is an $(n-1)$-subset S_1 of $V(K') - \{v_k, v_{k+1}\}$ such that $S_1 \cup \{v_{k+1}\} \in K'$ while $S_1 \cup \{v_k\} \notin K'$.

Let $T_1 = S_1 \cup \{v_1\}$ and observe that $T_1 \cup \{v_{k+1}\} \in K$ while $T_1 \cup \{v_k\} \notin K$. Since $d^n(v_i) = d'^{n-1}(v_i) + d''^n(v_i)$, $i = k, k+1$, and since $d_1, d_2, \dots, d_p$ is a decreasing sequence, it follows from the inequality $d'^{n-1}(v_k) < d'^{n-1}(v_{k+1})$ that $d''^n(v_k) > d''^n(v_{k+1})$. Accordingly, there is an n-subset T_2 of $V(K'') - \{v_k, v_{k+1}\}$ such that $T_2 \cup \{v_k\} \in K''$, while $T_2 \cup \{v_{k+1}\} \notin K''$. The situation is illustrated in Figure 1 below in dimension $n = 2$.

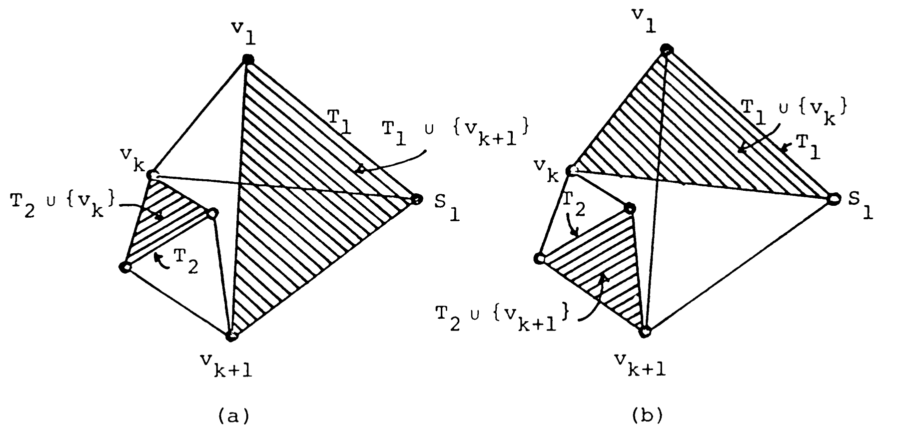

Figure 1. Before (a) and after (b) operating on K

In the above figure, the shaded triangles represent n-simplexes of K before and after the following operation is performed: remove the n-simplexes $T_2 \cup \{v_k\}$ and $T_1 \cup \{v_{k+1}\}$ from K and replace them by the n-simplexes $T_1 \cup \{v_k\}$ and $T_2 \cup \{v_{k+1}\}$. It is observed first that this operation leaves the numbers $d^n(v_i)$ unchanged since each vertex in the new complex (which we also denote by K) lies in the same number of n-simplexes as it did in the old complex K. Continue to take K' to be the union of $lk(v_1, K)$ with the set of 0-simplexes of K.

We find that all vertices have the same $(n-1)$ degree in the new complex K' as they did in the old with but two exceptions: $d'^{n-1}(v_k)$ has increased by unity, while $d'^{n-1}(v_{k+1})$ has decreased by unity.

Inspecting the formula for the slope of $v_2, v_3, \dots, v_p$,

$$\sum_{i=2}^{p} i \cdot d'^{n-1}(v_i) = \sum_{i=2}^{k-1} i \cdot d'^{n-1}(v_i) + k \cdot d'^{n-1}(v_k)$$

$$+ (k+1) \cdot d'^{n-1}(v_{k+1}) + \sum_{i=k+2}^{p} i \cdot d'^{n-1}(v_i),$$

we find the term $k \cdot d'^{n-1}(v_k)$ has increased by k, while the next term $(k+1) \cdot d'^{n-1}(v_{k+1})$ has decreased by $k+1$, a net decrease of unity. If the new sequence $d'^{n-1}(v_2), d'^{n-1}(v_3), \dots, d'^{n-1}(v_p)$ of $(n-1)$-degrees in the new complex K is decreasing, then the argument of Case 1 applies and the theorem follows. Otherwise a new integer k can be found such that $d'^{n-1}(v_k) < d'^{n-1}(v_{k+1})$ and a similar operation may be performed on the new K as was performed on the old K. Each time such an operation is carried out, the slope of $v_2, v_3, \ldots, v_p$ decreases by 1. Clearly, it cannot decrease below 0 and we are thus assured of sooner or later arriving at a complex K and subcomplex K' in which the sequence $d'^{n-1}(v_2), d'^{n-1}(v_3), \ldots, d'^{n-1}(v_p)$ is decreasing. This completes the proof of the theorem.

The following corollary is immediate.

Corollary. *The nonincreasing sequence $d_1, d_2, \ldots, d_p$ is n-graphical if and only if there is a decreasing $(n-1)$-graphical sequence $d'_1, d'_2, \ldots, d'_q$ for which the following conditions hold:*

(i) $\Sigma_{i=1}^{q} d'_i = (n-1)d_1$.

(ii) $d_2 - d'_1, d_3 - d'_2, \ldots, d_{q+1} - d'_q, d_{q+2}, \ldots, d_p$ *is n-graphical.*

It is interesting to observe how Theorem 2 specializes to graphs. When $n = 1$ in the statement of this theorem, the numbers $d'_1, d'_2, \ldots, d'_q$, are a (0, 0)-sequence such that $\Sigma_{i=1}^{p'} d'_i = d_i$. However, in a 0-complex, each vertex is incident with exactly one 0-simplex. Thus condition (ii) amounts to the subtraction of q 1's precisely as required in Havel's theorem. Condition (i) tells us that $q = d_1$.

As an illustration of the method implicit in Theorem 2, it may be decided whether the sequence 7, 7, 4, 3, 3, 3 is a (0, 2)-sequence or not. A decreasing (0, 1)-sequence $d'_1, d'_2, \ldots, d'_q$ having the property that $q \leq 5$ and $\Sigma_{i=1}^{q} d'_i = 2 \times 7 = 14$, is accordingly sought. There are only four (0, 1)-sequences having these properties and these are 4, 4, 2, 2, 2; 4, 3, 3, 3, 1; 4, 3, 3, 2, 2; 3, 3, 3, 3, 2. Subtracting these sequences in the manner of Theorem 2, one obtains the sequences 3, 0, 1, 1, 1; 3, 1, 0, 0, 2; 3, 1, 0, 1, 1; 4, 1, 0, 0, 1 and none of these is a (0, 2)-sequence. Therefore, by Theorem 2, the sequence 7, 7, 4, 3, 3, 3 is not a (0, 2)-sequence either.

On the other hand, the sequence 7, 6, 5, 3, 3, 3 is a (0, 2)-sequence: first, 4, 4, 2, 2, 2 is a (0, 1)-sequence and $4 + 4 + 2 + 2 + 2 = 2 \times 7 = 14$. Next, $6 - 4, 5 - 4, 3 - 2, 3 - 2, 3 - 2 = 2, 1, 1, 1, 1$ is a (0, 2)-sequence and Theorem 2 thus implies that 7, 6, 5, 3, 3, 3 is also a (0, 2)-sequence.

Theorem 2 contains two implicit procedures, one for discovering whether $d_1, d_2, \ldots, d_p$ is a $(0, n)$-sequence, the other for constructing an n-complex for which $d_1, \ldots, d_p$ is the decreasing n-degree sequence in case $d_1, d_2, \ldots, d_p$ happens to be a $(0, n)$-sequence.

3. **A problem.** It would be interesting to know whether there is a generalization of the characterization by P. Erdös and T. Gallai [2] of graphical degree sequences to $(0, n)$-sequences. Would any such generalization be as

complicated as the one we have found for Havel's characterization?

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