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CONTRACTED IDEALS AND PURITY FOR RING EXTENSIONS

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ABSTRACT. In this paper an example is given of a pair of commutative noetherian rings $R \subseteq S$ with S a finite R-module and $IS \cap R = I$ for each ideal I of R, but having the property that $0 \to R \to S$ is not a pure sequence of R-modules. Purity of the sequence $0 \to R \to S$ is equivalent to $R[X]$ being "ideally closed" in $S[X]$, X an indeterminate. Therefore, the example renders appealing the proposition that for R noetherian and S a noetherian torsion-free R-algebra containing R, if $\alpha S \cap R = \alpha R$ for each non-zero-divisor $\alpha \in R$, then the extension $R[X] \subseteq S[X]$ has the same properties. Finally, it is also shown that for R noetherian and $0 \to R \to S$ pure, with S an R-algebra, then $R[[X_1, \ldots, X_n]]$ is pure in $S[[X_1, \ldots, X_n]]$ for each positive integer n.

Let S be a ring extension of R with $R \subseteq S$. If X is an indeterminate, this note is concerned with the passage to the extension $R[X] \subseteq S[X]$ of certain properties of the extension $R \subseteq S$. Our motivation was the following problem: Suppose that R is noetherian. If for each ideal I of R, $IS \cap R = I$, does $R[X] \subseteq S[X]$ have the same property? Without the noetherian assumption, this question was known to have a negative answer and as we shall soon see, the answer is also negative for R noetherian. Happily, we salvage some positive results, the principal one being: Suppose that R is noetherian and that S is a torsion-free R-module. Then $S[X]$ is a torsion-free $R[X]$-module and if for each non-zero-divisor $\alpha \in R$, $\alpha S \cap R = \alpha R$, $R[X] \subseteq S[X]$ has the same property. The paper concludes with a brief treatment of these questions for the extension $R[[X]] \subseteq S[[X]]$.

We confine ourselves exclusively to commutative unitary rings, and for the most part we restrict our attention to noetherian rings. Our terminology is essentially that of [3].

It is convenient at this point to introduce the notion of purity for R-modules. An exact sequence $0 \longrightarrow E \longrightarrow F$ of R-modules is called a *pure* sequence if for each R-module G, $0 \longrightarrow E \otimes G \longrightarrow F \otimes G$ is exact. In our case, if $0 \longrightarrow R \longrightarrow S$ is a pure sequence we shall say that R is pure in S or that S is a pure extension of R. From Cohn's well-known solvability of linear equations criterion for purity [6, p. 65, Theorem 3.44], it follows immediately that if R is pure in S, then $IS \cap R = I$, for each ideal I of R. However,

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Enochs has given an example to show that the two conditions are not, in general, equivalent. His example appears in [1] and is mentioned in [2]. It is obvious from elementary properties of the tensor product that if R is pure in S, then $R[X]$ is pure in $S[X]$. Moreover, one of the key ideas occurring in [1] is the observation that R is pure in S if and only if for each positive integer n and each ideal I of $R[X_1, \ldots, X_n]$, the ideal $IS[X_1, \ldots, X_n] \cap R[X_1, \ldots, X_n] = I$, where $X_1, \ldots, X_n$ are indeterminates. Thus, by Hilbert's basis theorem for R noetherian, the condition $IS \cap R = I$ for each ideal I of R implies the same condition for $R[X] \subseteq S[X]$ if and only if $IS \cap R = I$ for each ideal I of R implies R is pure in S. The following example, suggested to us by E. L. Lady, shows that neither of these implications is valid.

Example 1. An artinian local ring R and an R-algebra S containing R, finite as an R-module, such that $IS \cap R = I$ for each ideal I of R, but R is not pure in S.

Let K be a finite field with q elements. Set

$$R = K[X, Y]/(X^{q+1}, Y^2, X^qY) = K[x, y]$$

and

$$S = R[T]/(T^2, xT - x^q, yT) = R[t].$$

Claim 1. $R \subseteq S$.

First of all, notice that $S = (R \oplus (R/yR)t)/((-x^q, xt))$, where $R \oplus (R/yR)t \simeq$ the idealization of the R-module R/yR [5, p. 2]. Now, to see that $R \subseteq S$, we need to see that $((-x^q, xt)) \cap R = (0)$. Thus, let $\alpha = (f(x, y), g(x)t)(-x^q, xt) \in R$. Then $\alpha = (-x^qf(x, y), -x^qg(x)t + xf(x, y)t)$. Suppose that $f(x, y) = a + m$, $a \in K$, $m \in (x, y)R$, the unique maximal ideal of R. $x^qm = 0$, so $-x^qf(x, y) = -ax^q$. Thus, if $a = 0$ and $\alpha \in R$, $\alpha = 0$. If $a \neq 0$ and $g(x) = b + xh(x)$, then

$$\beta = -x^qg(x)t + xf(x, y)t = -bx^qt + axt + x^2(f_1(x))t.$$

But $(R/yR)t \simeq K[X]/(X^{q+1})$ and since $a \neq 0$, $\beta \neq 0$. Thus, $\alpha \notin R$.

Claim 2. R is not pure in S.

Now $t \in S$ and is a solution of the system of linear equations

$$(*) \qquad xT = x^q, \qquad yT = 0.$$

This system has no solution in R, for suppose

$$f(x, y) = a + by + c_1x + c_2x^2 + \cdots + c_qx^q + d_1xy + d_2x^2y + \cdots + d_{q-1}x^{q-1}y$$

is a solution. Then

$$x^q = xf(x, y) = ax + bxy + c_1x^2 + \cdots + c_{q-1}x^q + d_1x^2y + \cdots + d_{q-2}x^{q-1}y,$$

and therefore $a = b = c_1 = \dots = c_{q-2} = d_1 = \dots = d_{q-2} = 0$, $c_{q-1} = 1$. Thus,

$$f(x, y) = x^{q-1} + c_q x^q + d_{q-1} x^{q-1} y.$$

But then $yf(x, y) = x^{q-1} y \neq 0$.

Claim 3. For each ideal I of R, $IS \cap R = I$.

It suffices to show that $(ax + by)t \in (ax + by)R$ for all $a, b \in R$. But $(ax + by)t = axt = ax^q$ and if $a = c + m$, $c \in K$, $m \in (x, y)$, then $ax^q = cx^q = c^q x^q = a^q x^q = (ax + by)^q$.

Claim 4. There exist principal ideals of $R[Z_1, Z_2]$ which are not contracted ideals from $S[Z_1, Z_2]$, Z_1, Z_2 indeterminates.

Since

$$t(xZ_1 + yZ_2) = txZ_1 + tyZ_2 = x^q Z_1,$$

$$x^q Z_1 \in ((xZ_1 + yZ_2)S[Z_1, Z_2]) \cap R[Z_1, Z_2].$$

Suppose that $x^q Z_1 \in (xZ_1 + yZ_2)R[Z_1, Z_2]$, say $x^q Z_1 = (F_0 + F_1 + \dots + F_m) \cdot (xZ_1 + yZ_2)$, F_i a form of degree i in $R[Z_1, Z_2]$. Clearly, we must have $x^q Z_1 = F_0(xZ_1 + yZ_2)$. But then $F_0 x = x^q$ and $F_0 y = 0$; that is, F_0 is a solution in R to the system $(*)$ of Claim 2. This was seen to be impossible.

We would like now to prove the result mentioned in the introduction so we require some terminology. An R-module E is called *torsion-free* if for each $r \in R$, $e \in E$, $re = 0$ implies r is a zero-divisor of R or $e = 0$. The condition that $R \subseteq S$ be a torsion-free ring extension is merely the condition that the total quotient ring of R is contained in the total quotient ring of S. We can now state our main theorem. Although not as strong as one would like, Claim 4 of the above example shows that it is the best possible.

Theorem 1. *Suppose that R is noetherian or an integral domain and that S is a torsion-free R-algebra, $S \supseteq R$. Then:*

(i) *$S[X]$ is a torsion-free $R[X]$-module.*

(ii) *If $\alpha S \cap R = \alpha R$ for each non-zero-divisor $\alpha \in R$, then $fS[X] \cap R[X] = fR[X]$ for each non-zero-divisor $f \in R[X]$.*

Proof. Before giving a proof which treats both cases simultaneously, we give an argument in the domain case which the reader would probably have discovered for himself.

Let $f \in R[X]$ be such that f is a zero-divisor in $S[X]$. Then there exists $d \in S$, $d \neq 0$ such that $df = 0$. Since R is a domain and S is torsion-free, $f = 0$. This proves (i). For (ii), let $f \in R[X]$ with $h \in fS[X] \cap R[X]$. Then $h = fg$, $g \in S[X]$. Assume the notation is such that

$$(a_m X^m + a_{m+1} X^{m+1} + \cdots)(b_0 + b_1 X + \cdots) = c_0 + c_1 X + \cdots,$$

where $a_i, c_i \in R$, $b_i \in S$, $a_m \neq 0$

and j is the smallest index so that $b_j \notin R$. Then $a_m b_j + a_{m+1} b_{j-1} + \cdots = c_{m+j}$ and so $a_m b_j \in a_m S \cap R = a_m R$. Therefore, $a_m b_j = a_m b_j'$, $b_j' \in R$. Cancelling a_m, we reach a contradiction and $g \in R[X]$.

(i) We have proved (i) when R is a domain, so assume now that R is noetherian. Let $f = \sum_{i=0}^n r_i X^i \in R[X]$ be a non-zero-divisor of $R[X]$. If $Z(R)$ denotes the zero-divisors of R, $Z(R)$ is a finite union of prime ideals of R, each of which is the annihilator of a single nonzero element of R. It follows that $(r_0, \ldots, r_n) \nsubseteq Z(R)$ since the annihilator of $(r_0, \ldots, r_n) = (0)$. Choose $r \in (r_0, \ldots, r_n) \backslash Z(R)$. If f is a zero-divisor in $S[X]$, then for some nonzero $s \in S$, sf and hence $sr = 0$. This is impossible.

(ii) Now suppose that $fg = h \in fS[X] \cap R[X]$, $f, h \in R[X]$, $g \in S[X]$, f a non-zero-divisor. Write $f = \sum_{i=0}^l r_i X^i$, $g = \sum_{i=0}^m s_i X^i$, $h = \sum_{i=0}^n t_i X^i$, where $n = l + m$, adding zero coefficients for t's if necessary. Then the equation $fg = h$ is equivalent to the system

$$(*) \qquad \sum_{i+j=k} r_i s_j = t_k, \qquad 0 \le k \le n.$$

Since f is a non-zero-divisor in $S[X]$, g is the unique solution of $h = fg$, and so $(s_0, \ldots, s_m)$ is the unique solution in S to $(*)$. Therefore, the homogeneous system associated with $(*)$ has no nontrivial solution. By McCoy's theorem [3, p. 147], the ideal I of R generated by the $(m+1) \times (m+1)$ subdeterminants of the coefficient matrix of $(*)$ has annihilator 0. When R is a domain, one of those determinants is nonzero, say d. By Cramer's rule, $ds_j \in R$ for each j, so $s_j \in R$ for each j and $g \in R[X]$. When R is noetherian, I contains a regular element $d \in R$. Again $ds_j \in R$ for each j and so $g \in R[X]$.

Note that (ii) can be interpreted as saying that for R noetherian and $R \subseteq S$ a torsion-free extension, $S \cap$ (total quotient ring of R) $= R$ passes to the polynomial extension $R[X] \subseteq S[X]$.

Note also that some hypothesis on R is required for (i). To see this, notice that Exercises 6 and 7 of [3, p. 62] give a ring R for which (a) nonzero-divisors are units and (b) there exist $p, q \in R$ such that the annihilator of $(p, q) = (0)$, $(p, q) \neq R$. Set $S = R[Y]/(pY, qY)$. Then $R \subseteq S$, S is torsion-free over R, $p + qX$ is a non-zero-divisor in $R[X]$, but in $S[X]$, $y(p + qX) = 0$. Here, y denotes the image of Y in S and clearly, $y \neq 0$.

Recall from [3] that the extension $R \subseteq S$ has (LO) for "lying over" if for each prime ideal P of R there exists a prime ideal P' of S such that $P' \cap R = P$. This is clearly equivalent to the condition that for each prime ideal P of R, $PS \cap R = P$. We are going to prove that no finiteness assumptions whatsoever are required to insure that $PS \cap R = P$, for each prime ideal P of R passes to polynomial extension. However, as Example 1 shows, even in case R is noetherian, knowing that $IS \cap R = I$ for *each* ideal I of R does not insure that for each irreducible ideal J of $R[X]$, $JS[X] \cap R[X] = J$.

This is because in the noetherian case knowing that irreducible ideals of

$R[X]$ are contracted from $S[X]$ is sufficient to guarantee that all ideals are contracted.

Proposition (*cf. McAdam* [4, *p.* 708, *Proposition* 1]). *If* $R \subseteq S$ *has* (LO), *then* $R[X] \subseteq S[X]$ *has* (LO).

Proof. Let P' be a prime ideal of $R[X]$, $P = P' \cap R$, and Q a prime ideal of S lying over P. If $P' = P[X]$, then $Q[X]$ lies over it. Otherwise, consider $\overline{P}'$ in $(R/P)[X]$. Set $D = R/P$, $D_1 = S/Q$, $K =$ quotient field of D, $L =$ quotient field of D_1. Then $\overline{P}'$ is a height one prime of $D[X]$ and $\overline{P}' \cap D = (0)$. Therefore $\overline{P}'K[X]$ is a nonzero prime of $K[X]$, say generated by f. It follows that $fL[X]$ is contained in a prime ideal M_1 of $L[X]$ and that $M_1 \cap K[X] = \overline{P}'K[X]$, so $M_1 \cap D[X] = \overline{P}'$. Let $\overline{Q}' = M_1 \cap D_1[X]$. Then $\overline{Q}' \cap D[X] = \overline{P}'$ and with $Q' =$ preimage of Q' in $S[X]$, we have $Q' \cap R[X] = P'$.

We now turn to the power series extension $R[[X]] \subseteq S[[X]]$. The fact that R pure in S implies $R[X]$ is pure in $S[X]$ is, as noted previously, merely a "change of rings" theorem. This technique does not apply to $R[[X]]$, and it is an open question whether R pure in S implies $R[[X]]$ pure in $S[[X]]$. We are able to settle the noetherian case and for this we require a a lemma.

For $\{X_\alpha\}$ a collection of indeterminates, denote by $\mathbf{X}$ the ideal of $R[[\{X_\alpha\}]]$ generated by the X_α's. Inspired by [2, p. 764, Theorem 8] we have

Lemma. *Suppose that* $R \subseteq S$. *Then* R *is pure in* S *if and only if for each finite family* $\{X_i\}$ *of indeterminates and for each ideal* I *of* $R[[\{X_i\}]]$, *the ideal* $IS[[\{X_i\}]] \cap R[[\{X_i\}]] \subseteq \bigcap_{k=0}^{\infty}(I + \mathbf{X}^k)$.

Proof. Let $r_{ij} \in R$, $r_j \in R$, $s_i \in S$ with $\sum_{i=1}^{n} r_{ij}s_i = r_j$, $j = 1, \ldots, m$. Let $X_1, \ldots, X_m$ be indeterminates and set $f_i = \sum_{j=1}^{m} r_{ij}X_j$ for $1 \le i \le n$ and $f = \sum_{j=1}^{m} r_jX_j$. Then $f = \sum_{j=1}^{n} s_if_i$. Set $I = (f_1, \ldots, f_n)R[[X_1, \ldots, X_n]]$. Thus,

$$f \in IS[[X_1, \ldots, X_n]] \cap R[[X_1, \ldots, X_n]] \subseteq \bigcap_{k=0}^{\infty} (I + \mathbf{X}^k) \subseteq I + \mathbf{X}^2$$

and so $f = \sum_{i=1}^{n} g_if_i + h$, $h \in \mathbf{X}^2$. But f is a form of degree 1, hence $f = \sum_{i=1}^{n} g_i(0)f_i$. It follows that $\sum_{i=1}^{n} r_{ij}g_i(0) = r_j$, $1 \le j \le m$, and by [6, p. 65, Theorem 3.44] R is pure in S.

The converse is proved in [2].

Note that it follows from Example 1 and the above Lemma that $IS \cap R = I$ for each ideal I of R does not pass to the power series extension $R[[X]] \subseteq S[[X]]$, even for R noetherian.

Theorem 2. *Let* $R \subseteq S$ *with* R *noetherian and* R *pure in* S. *Then for each positive integer* n, $R[[X_1, \ldots, X_n]]$ *is pure in* $S[[X_1, \ldots, X_n]]$.

Proof. By induction, it suffices to prove that $R[[X]]$ is pure in $S[[X]]$.

Let $\{X_i\}$ be a finite set of indeterminates distinct from X. In $R[[X, \{X_i\}]]$, set $\mathbf{X} = (X, \{X_i\})$ and $\mathbf{X}_0 = (\{X_i\})$. Since R is pure in S, by the Lemma, for each ideal I of $R[[X, \{X_i\}]] = (R[[X]])[[\{X_i\}]]$,

$$IS[[X, \{X_i\}]] \cap R[[X, \{X_i\}]] \subseteq \bigcap_{k=0}^{\infty} (I + \mathbf{X}^k) = I$$

since $(R[[X, \{X_i\}]], \mathbf{X})$ is a Zariski ring. But $\bigcap_{k=0}^{\infty}(I + \mathbf{X}_0^k) = I$ since $(R[[X, \{X_i\}]], \mathbf{X}_0)$ is also a Zariski ring. By the Lemma, $R[[X]]$ is pure in $S[[X]]$.

Added in proof. The question "Does R pure in S imply $R[[X]]$ pure in $S[[X]]$" has been settled in the negative.

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