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p-EXTREMAL LENGTH AND p-MEASURABLE CURVE FAMILIES

JOSEPH HESSE

ABSTRACT. It is well known that the reciprocal of p-extremal length, considered as a set function, is an outer measure. We show that if a curve family in euclidean n-space is measurable with respect to this outer measure, then the p-extremal length of the curve family is zero or infinite.

1. **Introduction.** Let $\mathcal{C}$ be the totality of curves in n-dimensional euclidean space R^n. More precisely, if $\gamma \in \mathcal{C}$ then γ is a continuous mapping of an open interval (a, b) into R^n where $a, b \in [-\infty, \infty]$. If $p \in (1, \infty)$ and $\Gamma \subset \mathcal{C}$, the p-extremal length of Γ, denoted by $\Lambda_p(\Gamma)$, is defined as follows. Let $\mathcal{F}(\Gamma)$ be the set of nonnegative Borel functions $\rho: R^n \to [0, \infty]$ satisfying the condition that for every locally rectifiable curve $\gamma \in \Gamma$, the line integral of ρ with respect to arc length $= \int_\gamma \rho\, ds \geq 1$. Then

$$\frac{1}{\Lambda_p(\Gamma)} = \inf \int_{R^n} \rho^p \, dm_n$$

where m_n is Lebesgue n-measure on R^n and the infimum is taken over all $\rho \in \mathcal{F}(\Gamma)$. If $\mathcal{F}(\Gamma) = \varnothing$ we set $\Lambda_p(\Gamma) = 0$. The p-modulus of Γ, denoted by $M_p(\Gamma)$, is defined as the reciprocal of the p-extremal length of Γ.

It is well known [1, Theorems 1 or 3, Theorem 6.2] that M_p, regarded as a set function on $\mathcal{C}$, is an outer measure. That is, 1. $M_p(\varnothing) = 0$, 2. $\Gamma_1 \subset \Gamma_2 \subset \mathcal{C}$ implies $M_p(\Gamma_1) \leq M_p(\Gamma_2)$, and 3. if $\Gamma_i \subset \mathcal{C}$, $i = 1, 2, \ldots$, then $M_p(\bigcup_{i=1}^\infty \Gamma_i) \leq \sum_{i=1}^\infty M_p(\Gamma_i)$. It appears reasonable to ask the question: What subsets of $\mathcal{C}$ are *p-measurable*? In other words, for which curve families $\Gamma \subset \mathcal{C}$ do we have

$$M_p(\theta) = M_p(\theta \cap \Gamma) + M_p(\theta - \Gamma) \tag{1.1}$$

for all $\theta \subset \mathcal{C}$? Renggli [2] has shown that if $p = 2$, $n = 2$, and $0 < M_2(\Gamma) < \infty$, then Γ is not 2-measurable. It is the purpose of this paper to simplify and generalize this result for all $p \in (1, \infty)$ and all $n \geq 2$. In particular, no use is made of conformal mappings.

2. **Preliminary lemmas.**

2.1. **Lemma.** *Let $E \subset R^{n-1}$ and $h \in (0, \infty)$. Let Γ be the family of curves $\gamma: (0, h) \to R^n$ such that for $t \in (0, h)$, $\gamma(t) = (x_1, \ldots, x_{n-1}, t)$*

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where $(x_1, \ldots, x_{n-1}) \in E$. *Then* $M_p(\Gamma) = m^*_{n-1}(E)/h^{p-1}$ *where* $m^*_{n-1}(E)$ *is the* $(n-1)$*-dimensional Lebesgue outer measure of* E. *Furthermore, if* E *is an* $(n-1)$*-Borel measurable set then the function* $\rho_0\colon R^n \to [0, \infty]$ *defined by*

$$\rho_0(x_1, \ldots, x_n) = \begin{cases} 1/h & \text{if } (x_1, \ldots, x_{n-1}) \in E \text{ and } 0 < x_n < h, \\ 0 & \text{otherwise,} \end{cases}$$

is in $\mathfrak{F}(\Gamma)$ *and* $M_p(\Gamma) = \int_{R^n} \rho_0^p \, dm_n$.

Proof. Let $V \subset R^{n-1}$ be an open set containing E. Let $\rho\colon R^n \to [0, \infty]$ be the Borel function defined by

$$\rho(x_1, \ldots, x_n) = \begin{cases} 1/h & \text{if } (x_1, \ldots, x_{n-1}) \in V \text{ and } 0 < x_n < h, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\rho \in \mathfrak{F}(\Gamma)$. Therefore $M_p(\Gamma) \le \int_{R^n} \rho^p \, dm_n \le m_{n-1}(V)/h^{p-1}$. Since this is valid for every open set $V \subset R^{n-1}$ which contains E, we get $M_p(\Gamma) \le m^*_{n-1}(E)/h^{p-1}$. We now proceed to establish the reverse inequality. Let $\rho \in \mathfrak{F}(\Gamma)$. Then $1 \le \int_0^h \rho(x_1, \ldots, x_{n-1}, t)\, dt$ for all $(x_1, \ldots, x_{n-1}) \in E$. An application of Hölder's inequality yields $h^{1-p} \le \int_0^h \rho^p(x_1, \ldots, x_{n-1}, t)\, dt$ for all $(x_1, \ldots, x_{n-1}) \in E$. Let F be the set of $(x_1, \ldots, x_{n-1}) \in R^{n-1}$ where $h^{1-p} \le \int_0^h \rho^p(x_1, \ldots, x_{n-1}, t)\, dt$. Then F is $(n-1)$-measurable and $E \subset F$. Therefore

$$\frac{m^*_{n-1}(E)}{h^{p-1}} \le \frac{m_{n-1}(F)}{h^{p-1}} \le \int_F \int_0^h \rho^p(x_1, \ldots, x_{n-1}, t)\, dt\, dm_{n-1} \le \int_{R^n} \rho^p \, dm_n.$$

Since $\rho \in \mathfrak{F}(\Gamma)$ is arbitrary, we get $m^*_{n-1}(E)/h^{p-1} \le M_p(\Gamma)$.

The other part of the lemma is clear.

2.2. **Comment.** In a way which can easily be made precise, the conclusions of the lemma remain correct if E, Γ and ρ_0 are "translated" by an amount $b \in R^n$.

The following lemma will be needed. The proof follows from [1, (e), p. 178] or [3, Theorem 6.7].

2.3. **Lemma.** *Let* $E_1, \ldots, E_k$ *be disjoint Borel sets in* R^n. *Let* $\Gamma_1, \ldots, \Gamma_k$ *be curve families such that every curve in* Γ_i *lies in* E_i, $i = 1, \ldots, k$. *Then* $M_p(\bigcup_{i=1}^k \Gamma_i) = \sum_{i=1}^k M_p(\Gamma_i)$.

3. Main lemma and theorem.

3.1. **Lemma.** *For every pair of positive integers* j, k *there exists a curve family* $\Gamma_{j,k} \subset \mathfrak{C}$ *with the following properties*: (i) $M_p(\Gamma_{j,k}) < \infty$, (ii) *every curve in* $\Gamma_{j,k}$ *lies in the closed cube* $S_k = \{(x_1, \ldots, x_n) \in R^n\colon |x_i| \le k\}$, (iii)

$$M_p(\Gamma_{j,k}) = \int_{R^n} (\rho_{j,k})^p \, dm_n \tag{3.2}$$

where $\rho_{j,k} \in \mathfrak{F}(\Gamma_{j,k})$ *is defined by*

$$\rho_{j,k}(x) = \begin{cases} j & \text{if } x \in S_k, \\ 0 & \text{if } x \in R^n - S_k, \end{cases} \tag{3.3}$$

and (iv) *if* $\Gamma \subset \mathcal{C}$ *then*

$$M_p(\Gamma_{j,k} \cap \Gamma) + (1 - 2^{1-p}) M_p(\Gamma_{2j,k} \cap \Gamma) \leq M_p(\Gamma). \tag{3.4}$$

Proof. Let $P^i_{j,k}$ be the bounded hyperplane defined by

$P^i_{j,k} = \{(x_1, \ldots, x_n) \in R^n : x_n = i/j\} \cap S_k, \qquad i = -kj, -(kj) + 1, \ldots, kj.$

Let $\Gamma^i_{j,k}$, defined for $i = -kj, \ldots, (kj) - 1$, be the family of curves γ: $(0, 1/j) \to S_k$ such that for $t \in (0, 1/j)$, $\gamma(t) = (x_1, \ldots, x_{n-1}, i/j + t)$ where $|x_l| \leq k$, $l = 1, \ldots, n-1$. $\Gamma^i_{j,k}$ is the family of segments in S_k which "connect" the planes $P^i_{j,k}$ and $P^{i+1}_{j,k}$ and which are parallel to the x_n-axis. Let $\Gamma_{j,k} = \bigcup_{i=-kj}^{kj-1} \Gamma^i_{j,k}$. Parts (i), (ii), and (iii) of the lemma follow from 2.1, 2.2, and 2.3.

We proceed to prove part (iv). Define $\Pi_j : S_k \to \bigcup_{i=-kj}^{kj-1} P^i_{j,k}$ as follows. If $(x_1, \ldots, x_n) \in S_k$ then for some $i \in \{-kj, \ldots, kj\}$ we have $i/j \leq x_n < (i+1)/j$. Set $\Pi_j(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, i/j)$. Let U be the set of points $\Pi_{2j,k}(x)$ where x is a point on a curve in $\Gamma_{2j,k} \cap \Gamma$. Lemma 2.1 gives

$$M_p(\Gamma_{2j,k} \cap \Gamma) = m^*_{n-1}(U)(2j)^{p-1}. \tag{3.5}$$

Let $\epsilon \in (0, 1)$ and choose a relatively open set V in $\bigcup_{i=-2kj}^{2kj-1} P^i_{2j,k}$ such that $U \subset V$ and

$$m_{n-1}(V) \leq m^*_{n-1}(U) + \epsilon. \tag{3.6}$$

Write $\Gamma_{j,k} \cap \Gamma = \alpha \cup \beta$ where β is the set of curves in $\Gamma_{j,k} \cap \Gamma$ which contain a point x such that $\Pi_{2j,k}(x) \in V$ and α is the complement of β in $\Gamma_{j,k} \cap \Gamma$. Lemma 2.1 gives

$$M_p(\beta) \leq m_{n-1}(V) j^{p-1}. \tag{3.7}$$

Since the curves in α and $\Gamma_{2j,k} \cap \Gamma$ lie in disjoint Borel sets, we conclude

$$M_p(\alpha) + M_p(\Gamma_{2j,k} \cap \Gamma) = M_p(\alpha \cup (\Gamma_{2j,k} \cap \Gamma)) \leq M_p(\Gamma).$$

Therefore,

$$M_p(\alpha) \leq M_p(\Gamma) - M_p(\Gamma_{2j,k} \cap \Gamma). \tag{3.8}$$

We also have

$$M_p(\Gamma_{j,k} \cap \Gamma) = M_p(\alpha \cup \beta) \leq M_p(\alpha) + M_p(\beta).$$

The above and relations 3.5–3.8 give

$$M_p(\Gamma_{j,k} \cap \Gamma) \leq M_p(\Gamma) - M_p(\Gamma_{2j,k} \cap \Gamma) + [M_p(\Gamma_{2j,k} \cap \Gamma)/(2j)^{p-1} + \epsilon]j^{p-1}.$$

Letting $\epsilon \to 0$ in the above and simplifying, we get 3.4.

3.9. **Theorem.** *Let* $\Gamma \subset \mathcal{C}$ *amd assume* $0 < M_p(\Gamma) < \infty$. *Then* Γ *is not* p-*measurable.*

Proof. We assume that Γ is p-measurable and proceed to obtain a contradiction.

Choose $\rho \in \mathcal{F}(\Gamma)$ such that $\int_{R^n} \rho^p \, dm_n < \infty$ and $\epsilon \in (0, 1)$. Let k be an integer such that $\int_{R^n - S_k} \rho^p \, dm_n < \epsilon/2$. Let j be an integer such that $\int_{\{x \in S_k : \rho(x) > j\}} \rho^p \, dm_n < \epsilon/2$. Let $\rho' = \max(\rho, \rho_{j,k})$. Then $\rho' \in \mathcal{F}(\Gamma \cup \Gamma_{j,k})$. Therefore,

$$\begin{aligned} M_p(\Gamma \cup \Gamma_{j,k}) - M_p(\Gamma_{j,k}) &\leq \int_{R^n} \rho'^p \, dm_n - \int_{R^n} \rho_{j,k}^p \, dm_n \\ &\leq \int_{\{x \in R^n : \rho(x) > \rho_{j,k}(x)\}} \rho^p \, dm_n \\ &\leq \int_{\{x \in S_k : \rho(x) > j\}} \rho^p \, dm_n + \int_{R^n - S_k} \rho^p \, dm_n \\ &\leq \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Therefore,

$$M_p(\Gamma \cup \Gamma_{j,k}) \leq M_p(\Gamma_{j,k}) + \epsilon. \tag{3.10}$$

In exactly the same way we get

$$M_p(\Gamma \cup \Gamma_{2j,k}) \leq M_p(\Gamma_{2j,k}) + \epsilon. \tag{3.11}$$

We now use the assumption of the p-measurability of Γ to get

$$M_p(\Gamma_{j,k}) = M_p(\Gamma_{j,k} \cap \Gamma) + M_p(\Gamma_{j,k} - \Gamma)$$

and

$$M_p(\Gamma_{j,k} \cup \Gamma) = M_p(\Gamma) + M_p(\Gamma_{j,k} - \Gamma).$$

The above equations imply

$$M_p(\Gamma_{j,k}) + M_p(\Gamma) = M_p(\Gamma_{j,k} \cup \Gamma) + M_p(\Gamma_{j,k} \cap \Gamma).$$

The above and 3.10 give

$$M_p(\Gamma) \leq M_p(\Gamma_{j,k} \cap \Gamma) + \epsilon. \tag{3.12}$$

In exactly the same way we get

$$M_p(\Gamma) \leq M_p(\Gamma_{2j,k} \cap \Gamma) + \epsilon. \tag{3.13}$$

Relations 3.4, 3.12, and 3.13 give

$$(M_p(\Gamma) - \epsilon) + (1 - 2^{1-p})(M_p(\Gamma) - \epsilon) \leq M_p(\Gamma).$$

Letting $\epsilon \to 0$ in the above allows us to conclude $M_p(\Gamma) \leq 0$. This contradicts the hypothesis that $0 < M_p(\Gamma)$.

4. Conclusion.

Theorem. *Let* $\Gamma \subset \mathcal{C}$. *Then* (a) *if* $M_p(\Gamma) = 0$, *then* Γ *is p-measurable*; (b) *if* $0 < M_p(\Gamma) < \infty$, *then* Γ *is not p-measurable*; *and* (c) *if* $M_p(\Gamma) = \infty$, *then* Γ *may or may not be p-measurable.*

Proof. (a) is obvious and (b) is Theorem 3.9.

If $M_p(\Gamma) = \infty$ and $0 < M_p(\mathcal{C} - \Gamma) < \infty$, then Γ is not p-measurable since the complement of a measurable set is measurable. If $M_p(\Gamma) = \infty$ and $M_p(\mathcal{C} - \Gamma) = 0$, then Γ is p-measurable since $\mathcal{C} - \Gamma$ is p-measurable and its complement is Γ. It is easy to construct examples to show that both of these possibilities can occur.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242