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ON THE CONVOLUTION OF A MEASURE AND A FUNCTION

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ABSTRACT. Complements to a theorem of Bourbaki on the convolution of a measure and a function.

The setting is as follows [3, Chapter VIII, §4, No. 1]: X is a locally compact space, G is a locally compact group acting continuously on the left in X, and β is a nonzero positive measure on X that is quasi-invariant under G; more precisely,

$$\gamma(s)\beta = \chi(s^{-1}, \cdot) \cdot \beta$$

for all $s \in G$, where χ is universally measurable and everywhere > 0 on $G \times X$.

The following proposition is central to the discussion of convolution of functions in [3]:

Proposition [3, Chapter VIII, §4, No. 1, Proposition 2]. *Let μ be a measure on G, f a locally β-integrable complex function on X. Assume that one of the following conditions is verified*:

(i) *f and χ are continuous*;

(ii) *G operates properly in X, and f is zero on the complement of a denumerable union of compact sets*;

(iii) *μ is carried by a denumerable union of compact sets.*

If μ and f are convolvable relative to β, then the function $s \mapsto f(s^{-1}x)\chi(s^{-1}, x)$ is essentially μ-integrable for locally β-almost all x; and if $\mu * f$ denotes any locally β-integrable function such that $(\mu * f) \cdot \beta = \mu * (f \cdot \beta)$, then

$$(\mu * f)(x) = \int f(s^{-1}x)\chi(s^{-1}, x)\, d\mu(s)$$

locally β-almost everywhere.

Case (ii) apparently needs an additional hypothesis, and the proof of Case (iii) given in [3] has some sizable gaps. The aim of this paper is to clarify these points; also, we reformulate condition (iii) so as to make it more flexible in applications, one of which is given. All notations and terminology are taken from [1]–[3].

By way of motivation, we remark that the roughly comparable result in

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the treatise of E. Hewitt and K. A. Ross is Lemma (20.6) of [5]. There, the function f is assumed to be a Borel function, therefore the function $(s, x) \mapsto f(s^{-1}x)$ is a Borel function on the product space. This smoothes the way for an application of Fubini's theorem [5, Theorem (13.9)]; moreover, all of the partial functions $s \mapsto f(s^{-1}x)$, $x \mapsto f(s^{-1}x)$ are Borel functions. Consequently, the measurability problem considered in Lemma 3 below does not arise in [5]; on the other hand, the Bourbaki formulation is more flexible in that non-Borel functions are allowed.

Lemma 1. *The following conditions on a measure μ are equivalent*:

(a) *μ is carried by a denumerable union of compact sets*;

(b) *μ is carried by a denumerable union of essentially μ-integrable sets*;

(c) *μ is carried by a μ-moderated, μ-measurable set*;

(d) *$|\mu| = \sum_1^\infty \mu_n$, where (μ_n) is a summable sequence of bounded positive measures.*

Proof. We can suppose $\mu \geq 0$. It is trivial that (a) implies (b).

(b) $\Rightarrow$ (d). Suppose μ is carried by $S = \bigcup_1^\infty S_n$, where the S_n are essentially μ-integrable and, as we may suppose, disjoint. The measures $\mu_n = \phi_{S_n} \cdot \mu$ (ϕ denotes characteristic function) are bounded [2, §5, No. 3, Corollary of Theorem 1]; since $\mu = \phi_S \cdot \mu$ and $\phi_S = \sum_1^\infty \phi_{S_n}$, it is an elementary consequence of the Lebesgue dominated convergence theorem that $\mu = \sum_1^\infty \mu_n$ [1, Chapter IV, §4, No. 3, Corollary 2 of Theorem 2], [2, §2, No. 1].

(d) $\Rightarrow$ (c). Suppose $\mu = \sum_1^\infty \mu_n$, where the μ_n are bounded positive measures. For each n, write $\mu_n = f_n \cdot \mu$ with f_n essentially μ-integrable [2, §5, No. 5, Theorem 2 and No. 3, Corollary of Theorem 1]. We can suppose that f_n is μ-integrable [2, §5, No. 3, Corollary 2 of Proposition 3]. Then for every n, the set $A_n = \{s: f_n(s) \neq 0\}$ is μ-measurable, and μ-moderated [2, §1, No. 3, Corollary of Proposition 9], hence so is $A = \bigcup_1^\infty A_n$. For all n, evidently $\phi_A \cdot \mu_n = \mu_n$, that is, $\complement A$ is locally μ_n-negligible; it follows that $\complement A$ is locally μ-negligible [2, §2, No. 2, Corollary 2 of Proposition 1], in other words A carries μ.

(c) $\Rightarrow$ (a). Suppose A is a μ-moderated, μ-measurable set such that $\phi_A \cdot \mu = \mu$. Then $A \subset \bigcup_1^\infty K_n \cup N$, where the K_n are compact and N is μ-negligible. Writing $S = \bigcup_1^\infty K_n$, the relation $\complement S \subset N \cup \complement A$ shows that $\complement S$ is locally μ-negligible.

Lemma 2. *If f is a locally β-integrable function on X and if μ is a measure on G such that μ and f are convolvable relative to β, then the function $F(s, x) = f(s^{-1}x)\chi(s^{-1}, x)$ on $G \times X$ is measurable for $\mu \otimes \beta$.*

Proof. For every $h \in \mathcal{K}(X)$, the function $(1 \otimes h)F$ is essentially integrable for $\mu \otimes \beta$ [3, Chapter VIII, §4, No. 1, proof of Proposition 2], hence

$\mu \otimes \beta$-measurable. Then F is $\mu \otimes \beta$-measurable by a routine application of the principle of localization [1, Chapter IV, §5, No. 2, Proposition 4]. Incidentally, in view of the hypotheses on χ, it is the same to say that the function $(s, x) \mapsto f(s^{-1}x)$ is $\mu \otimes \beta$-measurable.

Lemma 3. *With hypotheses as in Lemma 2, let* $M = \{x: F(\cdot, x)$ *is not* μ*-measurable*$\}$.

(1) *If* f *and* χ *are continuous, then* $M = \varnothing$.

(2) *If* F *(equivalently, the function* $(s, x) \mapsto f(s^{-1}x)$*) is moderated for* $\mu \otimes \beta$, *then* M *is* β*-negligible.*

(3) *If* μ *is carried by a denumerable union of compact sets, then* M *is locally* β*-negligible.*

Proof. (1) For every x, $F(\cdot, x)$ is continuous.

(2) In view of Lemma 2, this is immediate from [2, §8, No. 2, Proposition (2a)].

(3) In view of criterion (d) of Lemma 1, this follows from the proof of [2, §8, No. 2, Proposition (2b)].

Proof of the Proposition (under an added hypothesis in Case (ii)). We can suppose $f \geq 0$ and $\mu \geq 0$. Let $F(s, x) = f(s^{-1}x)\chi(s^{-1}, x)$, and let $g: X \to \overline{\mathbf{R}}_+$ be the function defined by the formula $g(x) = \int^{\bullet} F(\cdot, x)\, d\mu$. One has $g(x) = \int^{*} F(\cdot, x) d\mu$ in Case (i) (because every $F(\cdot, x)$ is continuous [2, §1, No. 1, Proposition 4]) and in Case (ii) (because, for each x, $F(\cdot, x)$ vanishes outside a denumerable union of compact sets [3, Chapter III, §4, No. 5, Theorem (1b)]).

Cases (i), (ii). As shown in [3], g is locally β-integrable and $g \cdot \beta = \mu * (f \cdot \beta)$, that is, g is a determination of $\mu * f$. In particular, the set $N = \{x: g(x) = +\infty\}$ is locally β-negligible. In Case (i), this means (in view of part (1) of Lemma 3) that $F(\cdot, x)$ is μ-integrable for locally β-almost all x. *In Case* (ii), *if one assumes that the function* $(s, x) \mapsto f(s^{-1}x)$ *is* $\mu \otimes \beta$*-moderated,* then it results from part (2) of Lemma 3 that $F(\cdot, x)$ is μ-integrable for locally β-almost all x.

Case (iii). Suppose μ is carried by a denumerable union S of compact sets. Since $\phi_S = 1$ locally μ-almost everywhere and S is μ-moderated, one has

$$g(x) = \int^{\bullet} F(\cdot, x)\phi_S\, d\mu = \int^{*} F(\cdot, x)\phi_S\, d\mu$$

for all $x \in X$. As shown in [3], g is again locally β-integrable and is a determination of $\mu * f$. In particular, $g(x) < +\infty$ locally β-a.e.; in view of part (3) of Lemma 3, this means that $F(\cdot, x)$ is essentially μ-integrable for locally β-almost all x [2, §1, No. 3, Proposition 9].

The following application is a slight extension of [3, Chapter VIII, §4, No. 5, Proposition 10]:

Corollary. *Let β be a relatively invariant, nonzero positive measure on G, and let f, g be locally β-integrable functions on G such that f and g are convolvable relative to β. If one of f, g is continuous or is zero outside a denumerable union of essentially β-integrable sets, then*

$$(f * g)(x) = \int g(s^{-1}x)f(s)\chi(s^{-1})d\beta(s) = \int f(xs^{-1})g(s)\chi'(s^{-1})\,d\beta(s)$$

for locally β-almost all x.

Proof. Here χ and χ' denote the left and right multiplicators of β, which are continuous [3, Chapter VII, §1, No. 8]. It is straightforward to show that the two essential integrals (or integrals) exist simultaneously and are then equal, thus it is immaterial whether the conditions are imposed on f or on g. We can suppose $f \geq 0$, $g \geq 0$. Let $\mu = f \cdot \beta$.

If g is continuous, one applies Case (i) of the Proposition to μ, g.

Suppose f is zero outside $S = \bigcup_1^\infty A_n$, where the A_n are essentially β-integrable. For each n, let h_n be a β-integrable function such that $\phi_{A_n} = h_n$ locally β-a.e. Let $B_n = \{x: h_n(x) = 1\}$. From $\phi_{B_n} = h_n\phi_{B_n}$, we see that B_n is β-integrable. Since $h_n^2 = h_n$ locally β-a.e., it results that $h_n = \phi_{B_n}$ locally β-a.e. (indeed, β-a.e. [2, §1, No. 3, Lemma 1]), therefore $\phi_{A_n} = \phi_{B_n}$ locally β-a.e. It follows easily that $S \subset B \cup N$, where $B = \bigcup_1^\infty B_n$ is β-moderated and N is locally β-negligible; one can even suppose that B is a denumerable union of compact sets [2, §1, No. 2, Proposition 5]. Since N is also locally negligible for $\mu = f \cdot \beta$ [2, §5, No. 5, Theorem 2], it follows that μ is carried by B; thus we are in the situation of Case (iii) of the Proposition.

Remark. In the Corollary, it also suffices that one of f, g be equal locally β-a.e. to a continuous function. More generally, suppose f, g, f', g' are locally β-integrable functions such that $f = f'$ locally β-a.e. and $g = g'$ locally β-a.e. It is elementary that if f and g are convolvable relative to β, then so are f' and g', and one then has $f * g = f' * g'$ locally β-a.e. Suppose, in addition, that $f * g$ has a determination h such that $h(x)$, for locally β-almost all x, is given by the (coexisting) integral formulas of the Corollary; for such an x, the first formula shows that f may be replaced by f' (in both formulas), the second that g may be replaced by g'; thus $f' * g'$ is also given by such formulas locally β-a.e.

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