PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

RATIONAL APPROXIMATION TO SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH ALGEBRAIC COEFFICIENTS[1]

WOLFGANG M. SCHMIDT

ABSTRACT. Let K be the field of formal series $\alpha = a_k t^k + \dots + a_0 + a_{-1}t^{-1} + \dots$ and let $|\ |$ be the valuation with $|\alpha| = 2^k$ if $a_k \neq 0$. Suppose $\alpha \in K$ satisfies an mth order linear differential equation whose coefficients are algebraic functions of t. Then for $\epsilon > 0$ there are only finitely many rational functions $p(t)/q(t)$ with $|\alpha - p(t)/q(t)| < |q(t)|^{-2-4m-\epsilon}$.

1. Let F be a field of characteristic zero and $F(t)$ the field of rational functions in the variable t with coefficients in F. The field $F(t)$ may be embedded in the field K of formal series

$$\alpha = a_k t^k + \dots + a_0 + a_{-1}t^{-1} + a_{-2}t^{-2} + \dots \tag{1}$$

with coefficients in F. In K one has the nonarchimedean valuation with $|\alpha| = 2^k$ if the leading coefficient in (1) is $a_k \neq 0$.

There is a power series version of Roth's theorem [2] on rational approximation to algebraic numbers (Uchiyama [5]): *If $\alpha \in K$ is algebraic over $F(t)$ and if $\epsilon > 0$, then there are at most finitely many rational functions $p(t)/q(t)$ with $|\alpha - (p(t)/q(t))| < |q(t)|^{-2-\epsilon}$.*

Denote the (formal) derivative of an element $\alpha \in K$ by α', and denote the higher derivatives by $\alpha^{(1)} = \alpha'$, $\alpha^{(2)}, \dots$.

Theorem. *Suppose $\alpha \in K$ satisfies a differential equation*

$$\beta_m \alpha^{(m)} + \dots + \beta_1 \alpha' + \beta_0 \alpha + \gamma = 0, \tag{2}$$

where $\beta_m \neq 0, \beta_{m-1}, \dots, \beta_0, \gamma$ are elements of K and are algebraic over $F(t)$. Then for any $\epsilon > 0$, at most finitely many rational functions $p(t)/q(t)$ satisfy

$$|\alpha - (p(t)/q(t))| < |q(t)|^{-2-4m-\epsilon}. \tag{3}$$

The case $m = 0$ reduces to Uchiyama's theorem. There is no reason to believe that the exponent $-2 - 4m - \epsilon$ in (3) is best possible if $m > 0$. The proof of the theorem will depend on a power series version of my generalization [3], [4] of Roth's theorem to simultaneous approximation. The same

Received by the editors September 13, 1974.
AMS (MOS) subject classifications (1970). Primary 10F45; Secondary 10F25.
[1] Written with partial support from NSF grant NSF-GP-33026X.

method could be used to prove results for other algebraic differential equations with algebraic coefficients.

Kolchin [1] proves a very general theorem on solutions of algebraic differential equations with coefficients in $F(t)$. He proves that $|\alpha - (p(t)/q(t))| \geq c(\alpha)|q(t)|^{-\delta}$, where δ is the "denomination" of the differential equation, i.e. the maximum of $i_0 + 2i_1 + \cdots + (m+1)i_m$ over all monomials $\alpha^{i_0}(\alpha^{(1)})^{i_1} \cdots (\alpha^{(m)})^{i_m}$ occurring with nonzero coefficient in the differential equation. Since the coefficients in (2) are not necessarily rational, Kolchin's theorem cannot be applied directly. But it is easily seen that every solution of (2) also satisfies certain algebraic differential equations with coefficients in $F(t)$: Suppose the coefficients $\beta_m, \ldots, \beta_0, \gamma$ of (2) generate a field L of degree d over $F(t)$. On the one hand, we can take the "norm" of (2) to obtain a differential equation of order m and of total degree d in $\alpha, \alpha', \ldots, \alpha^{(m)}$. This equation has denomination $\delta = (m+1)d$. On the other hand, if $\lambda_1, \ldots, \lambda_d$ is a basis of L over $F(t)$, then in view of (2) the functions λ_i $(1 \leq i \leq d)$ and $\lambda_i\alpha^{(j)}$ $(1 \leq i \leq d;\ j = 0, 1, \ldots)$ lie in a vector space of dimension $(m+1)d$ over $F(t)$. So $1, \alpha, \ldots, \alpha^{((m+1)d-1)}$ are linearly dependent over $F(t)$, and α satisfies a linear differential equation of order $(m+1)d - 1$ with coefficients in $F(t)$. This equation is independent of the particular solution α of (2), and it again has denomination $\delta = (m+1)d$. So Kolchin's theorem is stronger than ours if $(m+1)d \leq 4m + 2$.

I am indebted to the referee for valuable suggestions.

2. Divide the differential equation (2) by β_m and rewrite it as

$$\alpha^{(m)} + \beta_{0,m-1}\alpha^{(m-1)} + \cdots + \beta_{00}\alpha + \gamma_0 = 0. \tag{4}$$

Differentiation yields an equation of order $m+1$:

$$\alpha^{(m+1)} + \beta_{1m}\alpha^{(m)} + \cdots + \beta_{10}\alpha + \gamma_1 = 0.$$

Similarly, k differentiations give an equation of order $m+k$. In particular, α satisfies the following system of differential equations, consisting of (4) and m derivatives:

$$\begin{aligned}
D_0(\alpha) &= \qquad\qquad\qquad\qquad\quad \alpha^{(m)} + \cdots + \beta_{00}\alpha + \gamma_0 = 0,\\
D_1(\alpha) &= \qquad\qquad \alpha^{(m+1)} + \beta_{1m}\alpha^{(m)} + \cdots + \beta_{10}\alpha + \gamma_1 = 0,\\
&\cdots\\
D_m(\alpha) &= \alpha^{(2m)} + \beta_{m,2m-1}\alpha^{(2m-1)} + \cdots \qquad + \beta_{m0}\alpha + \gamma_m = 0.
\end{aligned}$$

We thus have

$$L_i(\alpha^{(2m)}, \ldots, \alpha', \alpha, 1) = 0 \qquad (i = 0, 1, \ldots, m), \tag{5}$$

where L_i is the linear form

$$L_i(X_{2m}, \ldots, X_1, X_0, Y) = X_{m+i} + \beta_{i,m+i-1}X_{m+i-1} + \cdots + \beta_{i0}X_0 + \gamma_i Y$$

in $n = 2m + 2$ variables $X_{2m}, \ldots, X_0, Y$.

Now suppose that $p(t)/q(t)$ satisfies (3). The ith derivative of $p(t)/q(t)$ is

$$(6) \qquad (p(t)/q(t))^{(i)} = p_i(t)/q(t)^{i+1} \qquad (i = 0, 1, \ldots)$$

with a certain polynomial $p_i(t)$. Since every $\kappa \in K$ has $|\kappa'| \leq |\kappa|$, inequality (3) yields

$$(7) \qquad |\alpha^{(i)} - (p(t)/q(t))^{(i)}| = |\alpha^{(i)} - (p_i(t)/q(t)^{i+1})| < |q(t)|^{-2-4m-\epsilon}.$$

Therefore by (5),

$$\begin{aligned} &|L_i(p_{2m}(t)/q(t)^{2m+1}, \ldots, p_0(t)/q(t), 1)| \\ &\quad = |L_i(p_{2m}(t)/q(t)^{2m+1}, \ldots, p_0(t)/q(t), 1) - L_i(\alpha^{(2m)}, \ldots, \alpha, 1)| \\ &\quad \leq c_1|q(t)|^{-2-4m-\epsilon} \qquad (i = 0, 1, \ldots, m), \end{aligned}$$

where c_1 is a constant which depends only on the coefficients of the linear forms $L_0, \ldots, L_m$, hence depends only on the given differential equation (2). Multiplying our inequalities by $|q(t)|^{2m+1}$ and introducing the polynomial vector

$$(8) \qquad \mathbf{r}(t) = (p_{2m}(t), p_{2m-1}(t)q(t), \ldots, p_0(t)q(t)^{2m}, q(t)^{2m+1})$$

with $n = 2m + 2$ components, we obtain

$$(9) \qquad |L_i(\mathbf{r}(t))| \leq c_1|q(t)|^{-1-2m-\epsilon} \qquad (i = 0, 1, \ldots, m).$$

It follows from (7) that

$$\begin{aligned} |p_i(t)| &\leq \max(|\alpha^{(i)}||q(t)|^{i+1}, |q(t)|^{i+1-2-4m-\epsilon}) \\ &\leq c_2|q(t)|^{i+1} \qquad (i = 0, 1, \ldots, m). \end{aligned}$$

Hence if the norm $\|\mathbf{r}(t)\|$ of $\mathbf{r}(t)$ is defined as the maximum of the valuations of its components, then

$$\|\mathbf{r}(t)\| \leq c_2|q(t)|^{2m+1}.$$

In conjunction with (9) this gives

$$(10) \qquad |L_i(\mathbf{r}(t))| \leq c_3\|\mathbf{r}(t)\|^{-1-\delta} \qquad (i = 0, 1, \ldots, m)$$

with $\delta = \epsilon/(2m + 1) > 0$.

The linear forms $L_0, \ldots, L_m$ are linearly independent. In fact the $n = 2m + 2$ linear forms $L_0, \ldots, L_m, L_{m+1}, \ldots, L_{2m+1}$ are linearly independent, where $L_{m+1} = X_{m-1}, \ldots, L_{2m} = X_0, L_{2m+1} = Y$. If $\|\mathbf{r}(t)\|$ is large, then (10) yields

$$|L_i(\mathbf{r}(t))| \leq \|\mathbf{r}(t)\|^{-1-(\delta/2)} \qquad (i = 0, 1, \ldots, m),$$

$$|L_i(\mathbf{r}(t))| \leq \|\mathbf{r}(t)\| \qquad (i = m+1, \ldots, 2m+1).$$

Putting $c_0 = \cdots = c_m = -1 - (\delta/2)$ and $c_{m+1} = \cdots = c_{2m+1} = 1$, we have

(11) $$c_0 + \cdots + c_{2m+1} < 0$$

and

(12) $$|L_i(\mathbf{r}(t))| \leq \|\mathbf{r}(t)\|^{c_i} \qquad (0 \leq i \leq 2m+1).$$

3. Now suppose for the moment that $L_0, \ldots, L_{2m+1}$ are $n = 2m+2$ linearly independent linear forms in n variables whose coefficients are real algebraic numbers. Suppose $c_0, \ldots, c_{2m+1}$ are constants with (11). Write $\mathbf{x} = (x_0, \ldots, x_{2m+1})$ for an n-tuple of rational integers, and further write $\|\mathbf{x}\| = \max(|x_0|, \ldots, |x_{2m+1}|)$. In the course of my generalization of Roth's theorem to simultaneous approximation, I proved the following ([4, Lemma 7]; it is an immediate consequence of [3, Corollary to Theorem 3]):

Subspace theorem. *There exist finitely many proper rational subspaces $S_1, \ldots, S_l$ of n-dimensional space, such that every integral solution $\mathbf{x}$ of $|L_i(\mathbf{x})| \leq \|\mathbf{x}\|^{c_i}$ $(0 \leq i \leq 2m+1)$ lies in one of these subspaces.*

Just as Roth's theorem has a power series analog (i.e. Uchiyama's theorem), so the subspace theorem has a power series analog.[2] There are finitely many $(n-1) = (2m+1)$-dimensional subspaces $S_1, \ldots, S_l$ such that every polynomial vector solution $\mathbf{r}(t)$ of (12) lies in one of these subspaces. These subspaces are rational in the sense that they are defined by a linear homogeneous equation whose coefficients are rational functions of t. Hence they are defined by an equation with polynomial coefficients.

Let S be one of these subspaces, defined by the equation

$$a_{2m}(t)x_{2m} + \cdots + a_0(t)x_0 + b(t)y = 0.$$

In view of (6), a vector $\mathbf{r}(t)$ of type (8) satisfies this equation precisely if

(13) $$a_{2m}(t)(p(t)/q(t))^{(2m)} + \cdots + a_0(t)(p(t)/q(t)) + b(t) = 0.$$

Thus $p(t)/q(t)$ satisfies a linear differential equation with polynomial coefficients.

Lemma. *Suppose the differential equation* (13) *has a rational solution $s_0(t) = p_0(t)/q_0(t)$. Then there is an N in $0 \leq N \leq 2m$ and there are rational functions $s_1(t), \ldots, s_N(t)$ such that every rational solution $s(t) = p(t)/q(t)$ of* (13) *is of the form*

[2] I have a graduate student working on this.

$$s(t) = s_0(t) + \sum_{i=1}^{N} c_i s_i(t)$$

with coefficients c_i *in* F.

Proof. Clearly, if both $s(t)$, $s_0(t)$ are solutions of the inhomogeneous equation (13), then $u(t) = s(t) - s_0(t)$ is a solution of the homogeneous differential equation

$$a_{2m}(t)u(t)^{(2m)} + \cdots + a_0(t)u(t) = 0. \tag{14}$$

Pick t_0 with $a_{2m}(t_0) \neq 0$, and consider at first (formal) power series solutions

$$u(t) = \sum_{n=0}^{\infty} b_n(t - t_0)^n$$

of (14). Given arbitrary $b_0, b_1, \ldots, b_{2m-1}$, the other coefficients can be computed one after another from (14). Thus the formal power series solutions of (14) form a vector space of dimension $2m$. Now if a rational solution $u(t)$ of (14) had a pole of order $p > 0$ at t_0, then $a_{2m}(t)u(t)^{(2m)}$ would have a pole of order $2m + p$, while the other summands in (14) would have a pole of lesser order. Therefore a rational solution of (14) has no pole at t_0, and is thus a power series in $t - t_0$. The rational solutions of (14) form a subspace of the space of power series solutions. This subspace has a dimension N with $0 \leq N \leq 2m$. Our Lemma is proved.

In particular, the denominators of the rational solutions of (13) have bounded degrees. So (3) has no solution with large $|q(t)|$. To finish the proof of the theorem it suffices to remark that for every given value of $|q(t)|$, there is at most one solution $p(t)/q(t)$ of (3). For if $p_1(t)/q_1(t) \neq p_2(t)/q_2(t)$ with $|q_1(t)| = |q_2(t)|$ both satisfied (3), we would have the impossible relation

$$\frac{1}{|q_1(t)|^2} \leq \frac{|q_1(t)p_2(t) - q_2(t)p_1(t)|}{|q_1(t)q_2(t)|} = \left|\frac{p_2(t)}{q_2(t)} - \frac{p_1(t)}{q_1(t)}\right| < |q_1(t)|^{-2-4m-\epsilon}$$

REFERENCES

1. E. R. Kolchin, *Rational approximation to solutions of algebraic differential equations*, Proc. Amer. Math. Soc. **10** (1959), 238–244. MR **21** #6366.

2. K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20; corrigendum, 168. MR **17**, 242.

3. W. M. Schmidt, *Linear forms with algebraic coefficients*. I, J. Number Theory **3** (1971), 253–277. MR **46** #7176.

4. ———, *Norm form equations*, Ann. of Math. (2) **96** (1972), 526–551. MR **47** #3313.

5. S. Uchiyama, *Rational approximations to algebraic functions*, J. Fac. Sci. Hokkaido Univ. Ser. I **15** (1961), 173–192. MR **24** #A1882.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80302