CHARACTERIZING A CIRCLE
WITH THE DOUBLE MIDSET PROPERTY

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ABSTRACT. A short and elementary proof is given to show that a
space $X$ is a circle with the natural geodesic metric if $X$ is a nonde-
generate, complete, convex metric space with the double midset property.

In 1970 Berard [1] announced that a complete convex metric space
having the double midset property must be a topological simple closed curve.
Based on a manuscript [2] subsequently received from him, Loveland and
Valentine [5] showed that under Berard's hypothesis the space is actually
isometric to a circle having the natural geodesic metric. Although Berard's
manuscript was not published, a paper by Berard and Nitka [3] has recently
appeared in which the isometry is established. However a short and elemen-
tary proof can be obtained by quoting Theorem 1 of [6] and Theorem 2 of [5],
with some extra work. Rather than proceed in this manner we have endeav-
or to make this note largely self-contained. Thus we adapt some proofs to
our situation and give them here.

The midset (called the "bisector" in [3]) of two points $a$ and $b$ of a
metric space $X$ is the set of all $x$ in $X$ such that the distances $ax$ and $bx$
are equal, and $X$ is said to have the double midset property (DMP) if, for
every pair of distinct points $a$ and $b$ of $X$, the midset $M(a, b)$ of $a$ and $b$
consists of two points.

In the remainder of the paper, $X$ will denote a nondegenerate, complete,
convex, metric space having the DMP. It is easy to see that the "complete,
convex" hypothesis can be replaced by "segment-convex" as is done in
[3]. The essential hypothesis is that $X$ contain with two of its points a
segment joining them (see [4, Theorem 14.1, p. 41]).

Lemma 1. The space $X$ contains a simple closed curve.

Proof. Let $a$ and $b$ be points of $X$, $S$ a segment with endpoints $a$ and
$b$, and $M(a, b) = \{m_1, m_2\}$. Since $S$ cannot have two midpoints, $m_2 \not\in S$. Let
$S'$ be the union of two segments $S_1$ and $S_2$ having endpoints $\{a, m_2\}$ and
$\{b, m_2\}$, respectively. Obviously $S \cup S'$ contains a simple closed curve
unless $S_1$ and $S_2$ share a segment $S_3$ with endpoints $m_1$ and $m_2$. However

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$S_3 \subset S_1 \cap S_2$ implies $S_3 \subset M(a, b)$, contradicting the DMP.

Lemma 2 [6, Theorem 1]. The space $X$ is a topological simple closed curve.

Proof. From Lemma 1, $X$ contains a simple closed curve $J$. Suppose there is a point $x$ in $X - J$. There cannot exist two points $a$ and $b$ of $J$ equidistant from $x$, for then $M(a, b)$ would contain $x$ and could intersect $J$ at most once. This would contradict the fact that $M(a, b)$ separates $a$ from $b$ in $X$. Thus the function $g: J \to R$, defined by $g(t) = xt$, is a continuous injection of $J$ into the real line. This is impossible since $g$ is a homeomorphism under these conditions.

Main theorem [5, Theorem 2]. The space $X$ is isometric to a circle having the natural geodesic metric.

Proof. Let $a$ and $x$ be two points of the simple closed curve $X$ (see Lemma 2), and let $S(a, b)$ be a maximal (with respect to inclusion) segment containing $x$ and having $a$ as an endpoint. Let $\{x_i\}$ be a monotone sequence of points of the open arc $X - S(a, b)$ converging to $b$. Since $S(a, b) \cap S(a, x_n) = \{a\}$, we see that the closure of $\bigcup_{i=1}^{\infty} S(a, x_i)$ is a segment $S'(a, b)$ and that $X = S(a, b) \cup S'(a, b)$. Let $C$ be a circle in $E^2$ of radius $ab/\pi$ with the geodesic metric. Let $f$ be a homeomorphism taking $X$ onto $C$ such that $f|S(a, b)$ and $f|S'(a, b)$ are both isometries onto semicircles of $C$. To show that $f$ is an isometry it suffices to check that $f(x)f(y) = xy$ whenever $x$ and $y$ are chosen in the interiors of $S(a, b)$ and $S'(a, b)$, respectively. We denote $f(z)$ by $z'$, and we define $pq$ to mean $pq + qr = pr$. We may assume that $xay$ holds. If $x' a' y'$ also holds, then $xy = x'y'$ as desired. Otherwise we have $x'b'y'$, and we now show this implies $xy$ from which $xy = x'y'$. Suppose $xb + by > xy$. Since $xy = xa + ay$, this leads to the contradiction that $x'y' > x'a' + a'y'$.

REFERENCES

2. ———, Characterizations of metric spaces by the use of their midsets: One-spheres (Unpublished manuscript, 1-14).