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CHARACTERIZING A CIRCLE WITH THE DOUBLE MIDSET PROPERTY

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ABSTRACT. A short and elementary proof is given to show that a space X is a circle with the natural geodesic metric if X is a nondegenerate, complete, convex metric space with the double midset property.

In 1970 Berard [1] announced that a complete convex metric space having the double midset property must be a topological simple closed curve. Based on a manuscript [2] subsequently received from him, Loveland and Valentine [5] showed that under Berard's hypothesis the space is actually isometric to a circle having the natural geodesic metric. Although Berard's manuscript was not published, a paper by Berard and Nitka [3] has recently appeared in which the isometry is established. However a short and elementary proof can be obtained by quoting Theorem 1 of [6] and Theorem 2 of [5], with some extra work. Rather than proceed in this manner we have endeavored to make this note largely self-contained. Thus we adapt some proofs to our situation and give them here.

The *midset* (called the "bisector" in [3]) of two points a and b of a metric space X is the set of all x in X such that the distances ax and bx are equal, and X is said to have the *double midset property* (DMP) if, for every pair of distinct points a and b of X, the midset $M(a, b)$ of a and b consists of two points.

In the remainder of the paper, X will denote a nondegenerate, complete, convex, metric space having the DMP. It is easy to see that the "complete, convex" hypothesis can be replaced by "segment-convex" as is done in [3]. The essential hypothesis is that X contain with two of its points a segment joining them (see [4, Theorem 14.1, p. 41]).

Lemma 1. *The space X contains a simple closed curve.*

Proof. Let a and b be points of X, S a segment with endpoints a and b, and $M(a, b) = \{m_1, m_2\}$. Since S cannot have two midpoints, $m_2 \notin S$. Let S' be the union of two segments S_1 and S_2 having endpoints $\{a, m_2\}$ and $\{b, m_2\}$, respectively. Obviously $S \cup S'$ contains a simple closed curve unless S_1 and S_2 share a segment S_3 with endpoints m_1 and m_2. However

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$S_3 \subset S_1 \cap S_2$ implies $S_3 \subset M(a, b)$, contradicting the DMP.

Lemma 2 [6, Theorem 1]. *The space X is a topological simple closed curve.*

Proof. From Lemma 1, X contains a simple closed curve J. Suppose there is a point x in $X - J$. There cannot exist two points a and b of J equidistant from x, for then $M(a, b)$ would contain x and could intersect J at most once. This would contradict the fact that $M(a, b)$ separates a from b in X. Thus the function $g: J \to R$, defined by $g(t) = xt$, is a continuous injection of J into the real line. This is impossible since g is a homeomorphism under these conditions.

Main theorem [5, Theorem 2]. *The space X is isometric to a circle having the natural geodesic metric.*

Proof. Let a and x be two points of the simple closed curve X (see Lemma 2), and let $S(a, b)$ be a maximal (with respect to inclusion) segment containing x and having a as an endpoint. Let $\{x_i\}$ be a monotone sequence of points of the open arc $X - S(a, b)$ converging to b. Since $S(a, b) \cap S(a, x_n) = \{a\}$, we see that the closure of $\bigcup_{i=1}^{\infty} S(a, x_i)$ is a segment $S'(a, b)$ and that $X = S(a, b) \cup S'(a, b)$. Let C be a circle in E^2 of radius ab/π with the geodesic metric. Let f be a homeomorphism taking X onto C such that $f|S(a, b)$ and $f|S'(a, b)$ are both isometries onto semicircles of C. To show that f is an isometry it suffices to check that $f(x)f(y) = xy$ whenever x and y are chosen in the interiors of $S(a, b)$ and $S'(a, b)$, respectively. We denote $f(z)$ by z', and we define pqr to mean $pq + qr = pr$. We may assume that xay holds. If $x'a'y'$ also holds, then $xy = x'y'$ as desired. Otherwise we have $x'b'y'$, and we now show this implies xby from which $xy = x'y'$. Suppose $xb + by > xy$. Since $xy = xa + ay$, this leads to the contradiction that $x'y' > x'a' + a'y'$.

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