PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

INVERSE CLUSTER SETS

T. R. HAMLETT AND PAUL E. LONG

ABSTRACT. For a function $f: X \to Y$, the cluster set of f at $x \in X$ is the set of all $y \in Y$ such that there exists a filter $\mathcal{F}$ on X converging to x and the filter generated by $f(\mathcal{F})$ converges to y. The inverse cluster set of f at $y \in Y$ is the set of all $x \in X$ such that y belongs to the cluster set of f at x. General properties of inverse cluster sets are proved, including a necessary and sufficient condition for continuity. Necessary and sufficient conditions for functions to have a closed graph in terms of inverse cluster sets are also given. Finally, a known theorem giving a condition as to when a connected function is also a connectivity function is generalized and further investigated in terms of inverse cluster sets.

1. **Introduction.** The idea of defining an inverse cluster set arises from the concept of a cluster set as found in [7] and [4] as well as elsewhere. The cluster set of a function $f: X \to Y$ at $x \in X$ is the set of all $y \in Y$ such that there exists a filter $\mathcal{F}$ on X converging to x and the filter generated by $f(\mathcal{F})$ converges to y. We define the inverse cluster set of f at $y \in Y$ to be the set of all $x \in X$ such that y belongs to the cluster set of f at x. After discussing some general properties of inverse cluster sets, their relationship to functions with closed graphs as well as connectedness is investigated.

Throughout, we use $\mathfrak{N}(x)$ to denote the neighborhood system at the point x. If $f: X \to Y$ is a function and $\mathcal{F}$ is a filter on X, the filterbase $f(\mathcal{F})$ generates a filter which we also denote by $f(\mathcal{F})$. The graph of a function $f: X \to Y$ is denoted by $G(f) = \{(x, f(x)): x \in X\}$. For the set A, $\mathrm{Cl}(A)$ denotes the closure of A.

2. **Basic properties of inverse cluster sets.**

2.1. **Definition** [4]. Let $f: X \to Y$ be any function. Then $y \in Y$ is an element of the *cluster set of f at x*, denoted by $\mathcal{C}(f; x)$, if there exists a filter $\mathcal{F}$ on X such that $\mathcal{F}$ converges to x and $f(\mathcal{F})$ converges to y.

2.2. **Definition.** Let $f: X \to Y$ be any function. The *inverse cluster set of f at $y \in Y$*, denoted by $\mathcal{C}^{-1}(f; y)$, is the set of all $x \in X$ such that $y \in \mathcal{C}(f; x)$.

2.3. **Theorem.** *Let $f: X \to Y$ be a function. Then the following are equivalent*:

Received by the editors December 9, 1974.

AMS (MOS) subject classifications (1970). Primary 54A20; Secondary 54C10.

(1) $x \in \mathcal{C}^{-1}(f; y)$.

(2) $x \in \bigcap\{\mathrm{Cl}(f^{-1}(V)): V \in \mathfrak{N}(y)\}$.

(3) *The filterbase* $f^{-1}(\mathfrak{N}(y))$ *accumulates to* x.

(4) $y \in \bigcap\{\mathrm{Cl}(f(U)): U \in \mathfrak{N}(x)\}$.

(5) $f(\mathfrak{N}(x))$ *accumulates to* y.

(6) *There exists a net* $x_\alpha \to x$ *such that* $f(x_\alpha) \to y$.

Proof. Theorem 2.2 of [4] states that $y \in \mathcal{C}(f; x)$ if and only if $f^{-1}(\mathfrak{N}(y))$ accumulates at x if and only if $y \in \bigcap\{\mathrm{Cl}(f(U)): U \in \mathfrak{N}(x)\}$. The conditions of the theorem then follow in a straightforward manner.

2.4. **Corollary.** *For* $f: X \to Y$, *the set* $\mathcal{C}^{-1}(f; y)$ *is closed for every* $y \in Y$.

2.5. **Corollary.** *If* $f: X \to Y$ *is a given function, then* $\mathrm{Cl}(f^{-1}(y)) \subset \mathcal{C}^{-1}(f; y)$ *for every* $y \in Y$.

Proof. Condition (2) of Theorem 2.3.

2.6. **Theorem.** *Let* X *be compact and* $f: X \to Y$ *a function such that* $\mathrm{Cl}(f(X)) = Y$. *Then* $\mathcal{C}^{-1}(f; y) \neq \emptyset$ *for every* $y \in Y$.

Proof. For each $y \in Y$, $f^{-1}(\mathfrak{N}(y))$ is a filterbase on the compact X, hence must have an accumulation point.

Evidently, the set $\mathcal{C}^{-1}(f; y)$ need not be connected even for continuous functions. In our efforts to find a condition under which $\mathcal{C}^{-1}(f; y)$ is connected, we use the following definition:

2.7. **Definition.** The function $f: X \to Y$ is *inverse connected* if $f^{-1}(C)$ is connected for every connected $C \subset Y$.

A sufficient condition for $f: X \to Y$ to be inverse connected, for example, is that f be closed and monotone [6, Theorem 2].

2.8. **Theorem.** *Let* $f: X \to Y$ *be inverse connected,* $\mathrm{Cl}(f(X)) = Y$, X *compact Hausdorff and* Y *locally connected. Then* $\mathcal{C}^{-1}(f; y)$ *is a nonempty continuum for every* $y \in Y$.

Proof. By Theorem 2.6, $\mathcal{C}^{-1}(f; y) \neq \emptyset$. Now let $\mathfrak{K}(y)$ be a neighborhood base of connected sets at y. It then follows from Theorem 2.3(2) that $\mathcal{C}^{-1}(f; y) = \bigcap\{\mathrm{Cl}(f^{-1}(V)): V \in \mathfrak{K}(y)\}$. Since $\{\mathrm{Cl}(f^{-1}(V)): V \in \mathfrak{K}(y)\}$ is a collection of continua directed by inclusion, their intersection is a continuum [8, Theorem 28.2].

2.9. **Theorem.** *Let* $f: X \to Y$ *be any function where* Y *is compact Hausdorff. Then* f *is continuous at* $x_0 \in X$ *if and only if* $x_0 \in \mathcal{C}^{-1}(f; y)$ *for exactly one* $y \in Y$.

Proof. Suppose first that f is continuous at x_0. Then $f(\mathfrak{N}(x_0)) \to f(x_0)$

and, since Y is Hausdorff, $f(\mathcal{N}(x_0))$ cannot accumulate to any other point.

Now suppose $x_0 \in \mathcal{C}^{-1}(f; y)$ for exactly one $y \in Y$. Assume f is not continuous at x_0. Then there exists an open V containing y such that $f(U) \cap (Y - V) \neq \varnothing$ for every $U \in \mathcal{N}(x_0)$. Thus, $f(\mathcal{N}(x_0))$ accumulates to some $y \in Y - V$ so that $y \in \mathcal{C}(f; x_0)$ which implies $x_0 \in \mathcal{C}^{-1}(f; y)$. But $x_0 \in \mathcal{C}^{-1}(f; f(x_0))$ also, and since $y \neq f(x_0)$, we have a contradiction to our hypothesis. It follows that f is continuous at x_0.

2.10. **Theorem** [4]. *Let $f: X \to Y$ be a connected function, X locally connected and Y compact Hausdorff. Then f is continuous at $x_0 \in X$ if and only if $\{y: x_0 \in \mathcal{C}^{-1}(f; y)\}$ is countable.*

2.11. **Theorem.** *Let $f: X \to Y$ be surjective. If $\mathcal{C}^{-1}(f; y)$ is degenerate for every $y \in Y$, then X is a T_1-space and f is a bijection.*

Proof. Theorem 2.4 shows each point in X is closed so that X is a T_1-space. If $f(x_1) = f(x_2) = y$, then $\{x_1, x_2\} \subset \mathcal{C}^{-1}(f; y)$. The hypothesis now implies $x_1 = x_2$ so that f is injective.

2.12. **Theorem.** *If $f: X \to Y$ is a bijection, then $\mathcal{C}^{-1}(f; y) = \mathcal{C}(f^{-1}; y)$.*

Proof. Definition 2.1 and Theorem 2.2 of [4] give $\mathcal{C}^{-1}(f; y) = \bigcap\{\mathrm{Cl}(f^{-1}(V)): V \in \mathcal{N}(y)\} = \mathcal{C}(f^{-1}; y)$.

2.13. **Theorem.** *Let $f: X \to Y$ be surjective and inverse connected where X is compact Hausdorff and Y is locally connected. If $\mathcal{C}^{-1}(f; y)$ is countable for every $y \in Y$, then*

(1) *f is a bijection, and*

(2) *f^{-1} is continuous.*

Proof. By Theorem 2.8, $\mathcal{C}^{-1}(f; y)$ is a nonempty continuum for every $y \in Y$ and the hypothesis that $\mathcal{C}^{-1}(f; y)$ is countable implies $\mathcal{C}^{-1}(f; y)$ is a singleton. It now follows from Theorem 2.11 that f is a bijection.

By Theorem 2.12, $\mathcal{C}^{-1}(f; y) = \mathcal{C}(f^{-1}; y)$ is degenerate for every $y \in Y$. Since X is compact Hausdorff, f^{-1} is continuous by Theorem 2.3 of [4].

3. Inverse cluster sets and the closed graph.

3.1. **Theorem.** *Let $f: X \to Y$ be any function and let $y \in Y$. Then $(x, y) \in X \times Y$ is a cluster point of $G(f)$ that does not belong to $G(f)$ if and only if $x \in \mathcal{C}^{-1}(f; y) - f^{-1}(y)$.*

3.2. **Theorem.** *For $f: X \to Y$, $G(f)$ is closed if and only if $\mathcal{C}^{-1}(f; y) = f^{-1}(y)$ for every $y \in Y$.*

Proof. Since $\mathcal{C}^{-1}(f; y) = f^{-1}(y)$ if and only if $\mathcal{C}^{-1}(f; y) - f^{-1}(y) = \varnothing$, Theorem 3.1 gives the desired result.

3.3. **Theorem.** *Let $f: X \to Y$ be closed and X regular. Then $\mathcal{C}^{-1}(f; y) = \mathrm{Cl}(f^{-1}(y))$ for every $y \in Y$.*

Proof. Since $\mathrm{Cl}(f^{-1}(y)) \subset \mathcal{C}^{-1}(f; y)$ for every $y \in Y$, we need only show the reverse inclusion. Suppose there exists a point $x \in X$ such that $x \in \mathcal{C}^{-1}(f; y) - \mathrm{Cl}(f^{-1}(y))$. The regularity of X then assures the existence of disjoint open sets U and V containing x and $\mathrm{Cl}(f^{-1}(y))$, respectively. Now using the fact that f is closed, there exists an open W containing y such that $f^{-1}(W) \subset V$ [2, Theorem 11.2, p. 86]. Hence, $x \notin \mathrm{Cl}(f^{-1}(W))$ which implies $x \notin \mathcal{C}^{-1}(f; y)$. This contradiction gives $\mathcal{C}^{-1}(f; y) \subset \mathrm{Cl}(f^{-1}(y))$ and establishes the theorem.

The following Corollary shows how one of Fuller's results [3] may be proved using inverse cluster sets.

3.4. **Corollary** [3, Corollary 3.9]. *Let $f: X \to Y$ be closed and X regular. If $f^{-1}(y)$ is closed for every $y \in Y$, then f has a closed graph.*

Proof. Theorems 3.3 and 3.2.

3.5. **Theorem.** *Let $f: X \to Y$ be closed and monotone where X is regular. Then $\mathcal{C}^{-1}(f; y)$ is connected for every $y \in Y$.*

Proof. By Theorem 3.3, $\mathcal{C}^{-1}(f; y) = \mathrm{Cl}(f^{-1}(y))$ for every $y \in Y$. Thus, $\mathcal{C}^{-1}(f; y)$ is the closure of the connected set $f^{-1}(y)$, hence connected.

We have seen that for a given function $f: X \to Y$, $\mathcal{C}^{-1}(f; y)$ is closed for every $y \in Y$. The following definition is used to determine a sufficient condition for a union of such sets to remain closed.

3.6. **Definition.** Let $f: X \to Y$ and let $A \subset Y$. Then $\mathcal{C}^{-1}(f; A) = \bigcup\{\mathcal{C}^{-1}(f; a): a \in A\}$.

3.7. **Theorem.** *Let $f: X \to Y$. If $A \subset Y$ is compact, then $\mathcal{C}^{-1}(f; A)$ is closed.*

Proof. First observe that

$$\mathcal{C}^{-1}(f; A) = \bigcup\{\mathcal{C}^{-1}(f; a): a \in A\} \subset \bigcap\{\mathrm{Cl}(f^{-1}(V)): V \text{ open and } A \subset V\}.$$

We now show the reverse inclusion. Let $x \in \bigcap\{\mathrm{Cl}(f^{-1}(V)): V$ open and $A \subset V\}$ and assume that for all $a \in A$ the filterbase $f^{-1}(\mathfrak{N}(a))$ does not accumulate to $x \in X$. Then for each $a \in A$ there exists a $V(a) \in \mathfrak{N}(a)$ and a $U_a \in \mathfrak{N}(x)$ such that $f^{-1}(V(a)) \cap U_a = \varnothing$. Now let $\{V(a_i): 1 \leq i \leq n\}$ be a finite subcollection of $\{V(a): a \in A\}$ which covers A and let $\{U_{a(i)}: 1 \leq i \leq n\}$ be the corresponding neighborhoods of x. It follows that

$$\bigcap\{U_{a(i)}: 1 \leq i \leq n\} \cap f^{-1}(\bigcup\{V(a_i): 1 \leq i \leq n\}) = \varnothing$$

so that $x \notin \bigcap\{\mathrm{Cl}(f^{-1}(V)): V$ open and $A \subset V\}$. But this contradicts our hypothesis. We conclude $x \in \mathcal{C}^{-1}(f; A)$ and this implies $\mathcal{C}^{-1}(f; A) = \bigcap\{\mathrm{Cl}(f^{-1}(V))\colon V$ open and $A \subset V\}$.

The following Corollary again shows how one of Fuller's results [3] may be obtained using inverse cluster sets.

3.8. **Corollary** [3, Theorem 3.6]. *Let $f\colon X \to Y$ be a given function with closed graph. If $A \subset Y$ is compact, then $f^{-1}(A)$ is closed.*

Proof. Theorem 3.2 along with Theorem 3.7 shows that

$$f^{-1}(A) = \bigcup\{f^{-1}(a)\colon a \in A\} = \bigcup\{\mathcal{C}^{-1}(f; a)\colon a \in A\} = \mathcal{C}^{-1}(f; A).$$

3.9. **Theorem.** *Let $f\colon X \to Y$ be continuous from the H-closed space X into the Hausdorff space Y. Then f maps regular-closed sets onto closed sets.*

Proof. Let M be a regular-closed subset of X. It follows that M is an H-closed subspace of X. Now consider any $y \in \mathrm{Cl}(f(M))$. Since $f^{-1}(\mathfrak{N}(y))$ is an open filterbase with a trace on M, $f^{-1}(\mathfrak{N}(y))$ accumulates to some $x \in M$ [1, Theorem 3.2] so that by Theorem 2.1(3), $x \in \mathcal{C}^{-1}(f; y)$. The fact that f has a closed graph, along with Theorem 2.2, implies $x \in \mathcal{C}^{-1}(f; y) = f^{-1}(y)$ so that $y \in f(M)$. We conclude $f(M)$ is closed.

4. **Connectivity functions and inverse cluster sets.** For a given function $f\colon X \to Y$ where both X and Y are first countable, the inverse cluster set $\mathcal{C}^{-1}(f; y)$ is precisely the set $T(f; y)$ as defined in [5, Definition 3.2]. We now show how cluster sets may be used to generalize Theorem 3.6 of [5] after recalling that a connected function $f\colon X \to Y$ is one that preserves connected sets and a connectivity function is one such that the induced function $g\colon X \to X \times Y$, defined by $g(x) = (x, f(x))$, is connected.

4.1. **Theorem.** *Let $f\colon X \to Y$ be a connected function where X is compact. Then f is a connectivity function if for each connected $M \subset X$ and any $x \in M$, $\mathcal{C}^{-1}(f; f(x)) \cap \mathrm{Cl}(M) = \{x\}$.*

Proof. Let f be connected and assume the given condition. Suppose there exists a connected $M \subset X$ such that $g(M) = H \cup K$ where H and K are separated and define $A = g^{-1}(H) \cap M$ and $B = g^{-1}(K) \cap M$. Then for any $x \in A$, there exist open sets $U \in \mathfrak{N}(x)$ and $V \in \mathfrak{N}(f(x))$ such that $(U \times V) \cap K = \varnothing$. Consequently, no point of $U \cap B$ can map into V under f. Since $f(M) = f(A) \cup f(B)$ and neither $f(A)$ nor $f(B)$ can be empty, we proceed to show $f(A)$ and $f(B)$ are separated, thereby obtaining a contradiction. Suppose $f(x) \in \mathrm{Cl}(f(B))$ for $x \in A$. Then $f^{-1}(\mathfrak{N}(y)) \cap B$ is a filterbase on the compact set $\mathrm{Cl}(M)$ and, hence, accumulates to some $x_0 \in \mathrm{Cl}(M)$. Therefore, $x_0 \in \mathcal{C}^{-1}(f, f(x))$, and, since $f(U \cap B) \cap V = \varnothing$, we have $\mathrm{Cl}(f^{-1}(V) \cap B) \cap U = \varnothing$. Consequently, $\{x, x_0\} \subset \mathcal{C}^{-1}(f, f(x)) \cap \mathrm{Cl}(M)$ which contradicts the given condition of the theorem. We conclude $f(x) \notin \mathrm{Cl}(f(B))$ for every $x \in A$ and, likewise, $f(x) \notin \mathrm{Cl}(f(A))$ for every $x \in B$. This implies $f(M)$ is not connected. Since f is given as a connected function, it must follow that g is connected.

The following Lemma and Theorem give a more workable insight into the condition stated in Theorem 4.1.

4.2. **Lemma.** *Let $f: X \to Y$ be a given function and let $A \subset X$. If $\mathcal{C}^{-1}(f|A; y)$ denotes the inverse cluster set of $f|A: A \to Y$ where A has the subspace topology, then $\mathcal{C}^{-1}(f|A; y) \subset \mathcal{C}^{-1}(f; y) \cap A$ and the equality holds provided A is open.*

Proof. The proof consists of the following set relationships:

$$\begin{aligned}\mathcal{C}^{-1}(f|A;\, y) &= \bigcap\{\mathrm{Cl}_A((f|A)^{-1}(V)) \colon V \in \mathcal{N}(y)\}\\ &= \bigcap\{\mathrm{Cl}_A(f^{-1}(V) \cap A) \colon V \in \mathcal{N}(y)\}\\ &\subset \bigcap\{\mathrm{Cl}(f^{-1}(V)) \cap A \colon V \in \mathcal{N}(y)\}\\ &= \bigcap\{\mathrm{Cl}(f^{-1}(V)) \colon V \in \mathcal{N}(y)\} \cap A\\ &= \mathcal{C}^{-1}(f;\, y) \cap A.\end{aligned}$$

Observe that if A is open, the subset relation in the proof is an equality.

4.3. **Theorem.** *Let $f: X \to Y$ be a given function and consider the following conditions:*

(1) *For each connected set $M \subset X$ and $x \in M$, $\mathcal{C}^{-1}(f; f(x)) \cap \mathrm{Cl}(M) = \{x\}$.*

(2) *For each component C of X, $f|C: C \to f(C)$ is a bijection with a closed graph.*

Then (1) *implies* (2) *and if X is locally connected,* (1) *and* (2) *are equivalent.*

Proof. To show (1) implies (2), let C be a component of X. Then for each $x \in C$, where $y = f(x)$, we have by Lemma 4.2 and the fact that C is closed,

$$\mathcal{C}^{-1}(f|C;\, y) \subset \mathcal{C}^{-1}(f;\, y) \cap C = \mathcal{C}^{-1}(f;\, y) \cap \mathrm{Cl}(C) = \{x\}.$$

Thus, $f|C$ is a bijection by Theorem 2.11. Since $\mathcal{C}^{-1}(f|C; y) = f^{-1}(y)$, Theorem 3.2 gives the graph of f closed.

Now assume (2) holds where X is locally connected, $M \subset X$ is connected and $x \in M$. Let C be the component of X containing $\mathrm{Cl}(M)$ and recall that components of locally connected spaces are open so that Lemma 4.2 holds. Then we have

$$\mathcal{C}^{-1}(f;\, f(x)) \cap \mathrm{Cl}(M) \subset \mathcal{C}^{-1}(f;\, f(x)) \cap C = \mathcal{C}^{-1}(f\,|C;\, f(x)) = \{x\}.$$

REFERENCES

1. M. P. Berri, J. R. Porter and R. M. Stephenson, Jr., *A survey of minimal topological spaces*, General Topology and its Relations to Modern Analysis and Algebra, III (Proc. Conf., Kanpur, 1968), Academia, Prague, 1971, pp. 93–114. MR 43 #3985.

2. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966.

3. R. V. Fuller, *Relations among continuous and various non-continuous functions*, Pacific J. Math. 25 (1968), 495–509. MR 37 #3536.

4. T. R. Hamlett, *Cluster sets in general topology*, J. London Math. Soc. (to appear).

5. Paul E. Long, *Connected mappings*, Duke Math. J. 35 (1968), 677–682. MR 38 #2745.

6. ———, *Concerning semiconnected maps*, Proc. Amer. Math. Soc. 21 (1969), 117–118. MR 38 #5183.

7. J. D. Weston, *Some theorems on cluster sets*, J. London Math. Soc. 33 (1958), 435–441. MR 20 #7109.

8. Stephen Willard, *General topology*, Addison-Wesley, Reading, Mass., 1970. MR 41 #9173.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, ARKANSAS 72701