INVERSE CLUSTER SETS

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ABSTRACT. For a function \( f: X \to Y \), the cluster set of \( f \) at \( x \in X \) is the set of all \( y \in Y \) such that there exists a filter \( J \) on \( X \) converging to \( x \) and the filter generated by \( f(J) \) converges to \( y \). The inverse cluster set of \( f \) at \( y \in Y \) is the set of all \( x \in X \) such that \( y \) belongs to the cluster set of \( f \) at \( x \). General properties of inverse cluster sets are proved, including a necessary and sufficient condition for continuity. Necessary and sufficient conditions for functions to have a closed graph in terms of inverse cluster sets are also given. Finally, a known theorem giving a condition as to when a connected function is also a connectivity function is generalized and further investigated in terms of inverse cluster sets.

1. Introduction. The idea of defining an inverse cluster set arises from the concept of a cluster set as found in [7] and [4] as well as elsewhere. The cluster set of a function \( f: X \to Y \) at \( x \in X \) is the set of all \( y \in Y \) such that there exists a filter \( \mathcal{F} \) on \( X \) converging to \( x \) and the filter generated by \( f(\mathcal{F}) \) converges to \( y \). We define the inverse cluster set of \( f \) at \( y \in Y \) to be the set of all \( x \in X \) such that \( y \) belongs to the cluster set of \( f \) at \( x \).

After discussing some general properties of inverse cluster sets, their relationship to functions with closed graphs as well as connectedness is investigated.

Throughout, we use \( \mathcal{N}(x) \) to denote the neighborhood system at the point \( x \). If \( f: X \to Y \) is a function and \( \mathcal{F} \) is a filter on \( X \), the filterbase \( f(\mathcal{F}) \) generates a filter which we also denote by \( f(\mathcal{F}) \). The graph of a function \( f: X \to Y \) is denoted by \( G(f) = \{(x, f(x)) : x \in X\} \). For the set \( A \), \( \text{Cl}(A) \) denotes the closure of \( A \).

2. Basic properties of inverse cluster sets.

2.1. Definition [4]. Let \( f: X \to Y \) be any function. Then \( y \in Y \) is an element of the cluster set of \( f \) at \( x \), denoted by \( C(f; x) \), if there exists a filter \( \mathcal{F} \) on \( X \) such that \( \mathcal{F} \) converges to \( x \) and \( f(\mathcal{F}) \) converges to \( y \).

2.2. Definition. Let \( f: X \to Y \) be any function. The inverse cluster set of \( f \) at \( y \in Y \), denoted by \( C^{-1}(f; y) \), is the set of all \( x \in X \) such that \( y \in C(f; x) \).

2.3. Theorem. Let \( f: X \to Y \) be a function. Then the following are equivalent:

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(1) \( x \in \mathcal{C}^{-1}(f; y) \).
(2) \( x \in \bigcap \{ \text{Cl}(f^{-1}(V)); V \in \mathcal{N}(y) \} \).
(3) The filterbase \( f^{-1}(\mathcal{N}(y)) \) accumulates to \( x \).
(4) \( y \in \bigcap \{ \text{Cl}(f(U)); U \in \mathcal{N}(x) \} \).
(5) \( f(\mathcal{N}(x)) \) accumulates to \( y \).
(6) There exists a net \( x_a \to x \) such that \( f(x_a) \to y \).

Proof. Theorem 2.2 of [4] states that \( y \in \mathcal{C}(f; x) \) if and only if \( f^{-1}(\mathcal{N}(y)) \) accumulates at \( x \) if and only if \( y \in \bigcap \{ \text{Cl}(f(U)); U \in \mathcal{N}(x) \} \). The conditions of the theorem then follow in a straightforward manner.

2.4. Corollary. For \( f: X \to Y \), the set \( \mathcal{C}^{-1}(f; y) \) is closed for every \( y \in Y \).

2.5. Corollary. If \( f: X \to Y \) is a given function, then \( \text{Cl}(f^{-1}(y)) \subseteq \mathcal{C}^{-1}(f; y) \) for every \( y \in Y \).

Proof. Condition (2) of Theorem 2.3.

2.6. Theorem. Let \( X \) be compact and \( f: X \to Y \) a function such that \( \text{Cl}(f(X)) = Y \). Then \( \mathcal{C}^{-1}(f; y) \neq \emptyset \) for every \( y \in Y \).

Proof. For each \( y \in Y \), \( f^{-1}(\mathcal{N}(y)) \) is a filterbase on the compact \( X \), hence must have an accumulation point.

Evidently, the set \( \mathcal{C}^{-1}(f; y) \) need not be connected even for continuous functions. In our efforts to find a condition under which \( \mathcal{C}^{-1}(f; y) \) is connected, we use the following definition:

2.7. Definition. The function \( f: X \to Y \) is inverse connected if \( f^{-1}(C) \) is connected for every connected \( C \subseteq Y \).

A sufficient condition for \( f: X \to Y \) to be inverse connected, for example, is that \( f \) be closed and monotone [6, Theorem 2].

2.8. Theorem. Let \( f: X \to Y \) be inverse connected, \( \text{Cl}(f(X)) = Y \), \( X \) compact Hausdorff and \( Y \) locally connected. Then \( \mathcal{C}^{-1}(f; y) \) is a nonempty continuum for every \( y \in Y \).

Proof. By Theorem 2.6, \( \mathcal{C}^{-1}(f; y) \neq \emptyset \). Now let \( K(y) \) be a neighborhood base of connected sets at \( y \). It then follows from Theorem 2.3(2) that \( \mathcal{C}^{-1}(f; y) = \bigcap \{ \text{Cl}(f^{-1}(V)); V \in K(y) \} \). Since \( \{ \text{Cl}(f^{-1}(V)); V \in K(y) \} \) is a collection of continua directed by inclusion, their intersection is a continuum [8, Theorem 28.2].

2.9. Theorem. Let \( f: X \to Y \) be any function where \( Y \) is compact Hausdorff. Then \( f \) is continuous at \( x_0 \in X \) if and only if \( x_0 \in \mathcal{C}^{-1}(f; y) \) for exactly one \( y \in Y \).

Proof. Suppose first that \( f \) is continuous at \( x_0 \). Then \( f(\mathcal{N}(x_0)) \to f(x_0) \).
and, since $Y$ is Hausdorff, $f(\mathcal{H}(x_0))$ cannot accumulate to any other point.

Now suppose $x_0 \in \mathcal{C}^{-1}(f; y)$ for exactly one $y \in Y$. Assume $f$ is not continuous at $x_0$. Then there exists an open $V$ containing $y$ such that $f(U) \cap (Y - V) \neq \emptyset$ for every $U \in \mathcal{H}(x_0)$. Thus, $f(\mathcal{H}(x_0))$ accumulates to some $y \in Y - V$, so that $y \in \mathcal{C}(f; x_0)$ which implies $x_0 \in \mathcal{C}^{-1}(f; y)$. But $x_0 \in \mathcal{C}^{-1}(f; f(x_0))$ also, and since $y \neq f(x_0)$, we have a contradiction to our hypothesis. It follows that $f$ is continuous at $x_0$.

2.10. Theorem [4]. Let $f: X \to Y$ be a connected function, $X$ locally connected and $Y$ compact Hausdorff. Then $f$ is continuous at $x_0 \in X$ if and only if $\bigl\{ y: x_0 \in \mathcal{C}^{-1}(f; y)\bigr\}$ is countable.

2.11. Theorem. Let $f: X \to Y$ be surjective. If $\mathcal{C}^{-1}(f; y)$ is degenerate for every $y \in Y$, then $X$ is a $T_1$-space and $f$ is a bijection.

Proof. Theorem 2.4 shows each point in $X$ is closed so that $X$ is a $T_1$-space. If $f(x_1) = f(x_2) = y$, then $\{x_1, x_2\} \subset \mathcal{C}^{-1}(f; y)$. The hypothesis now implies $x_1 = x_2$ so that $f$ is injective.

2.12. Theorem. If $f: X \to Y$ is a bijection, then $\mathcal{C}^{-1}(f; y) = \mathcal{C}(f^{-1}; y)$.

Proof. Definition 2.1 and Theorem 2.2 of [4] give $\mathcal{C}^{-1}(f; y) = \bigcap \{ \mathcal{C}(f^{-1}(V)) : V \in \mathcal{H}(y) \} = \mathcal{C}(f^{-1}; y)$.

2.13. Theorem. Let $f: X \to Y$ be surjective and inverse connected where $X$ is compact Hausdorff and $Y$ is locally connected. If $\mathcal{C}^{-1}(f; y)$ is countable for every $y \in Y$, then

1. $f$ is a bijection, and
2. $f^{-1}$ is continuous.

Proof. By Theorem 2.8, $\mathcal{C}^{-1}(f; y)$ is a nonempty continuum for every $y \in Y$ and the hypothesis that $\mathcal{C}^{-1}(f; y)$ is countable implies $\mathcal{C}^{-1}(f; y)$ is a singleton. It now follows from Theorem 2.11 that $f$ is a bijection.

By Theorem 2.12, $\mathcal{C}^{-1}(f; y) = \mathcal{C}(f^{-1}; y)$ is degenerate for every $y \in Y$. Since $X$ is compact Hausdorff, $f^{-1}$ is continuous by Theorem 2.3 of [4].

3. Inverse cluster sets and the closed graph.

3.1. Theorem. Let $f: X \to Y$ be any function and let $y \in Y$. Then $(x, y) \in X \times Y$ is a cluster point of $G(f)$ that does not belong to $G(f)$ if and only if $x \in \mathcal{C}^{-1}(f; y) - f^{-1}(y)$.

3.2. Theorem. For $f: X \to Y$, $G(f)$ is closed if and only if $\mathcal{C}^{-1}(f; y) = f^{-1}(y)$ for every $y \in Y$.

Proof. Since $\mathcal{C}^{-1}(f; y) = f^{-1}(y)$ if and only if $\mathcal{C}^{-1}(f; y) - f^{-1}(y) = \emptyset$, Theorem 3.1 gives the desired result.
3.3. Theorem. Let \( f: X \to Y \) be closed and \( X \) regular. Then \( C^{-1}(f; y) = \text{Cl}(f^{-1}(y)) \) for every \( y \in Y \).

Proof. Since \( \text{Cl}(f^{-1}(y)) \subseteq C^{-1}(f; y) \) for every \( y \in Y \), we need only show the reverse inclusion. Suppose there exists a point \( x \in X \) such that \( x \in C^{-1}(f; y) \setminus \text{Cl}(f^{-1}(y)) \). The regularity of \( X \) then assures the existence of disjoint open sets \( U \) and \( V \) containing \( x \) and \( \text{Cl}(f^{-1}(y)) \), respectively. Now using the fact that \( f \) is closed, there exists an open \( W \) containing \( y \) such that \( f^{-1}(W) \subseteq V \) [2, Theorem 11.2, p. 86]. Hence, \( x \notin \text{Cl}(f^{-1}(W)) \) which implies \( x \notin C^{-1}(f; y) \). This contradiction gives \( C^{-1}(f; y) \subseteq \text{Cl}(f^{-1}(y)) \) and establishes the theorem.

The following Corollary shows how one of Fuller's results [3] may be proved using inverse cluster sets.

3.4. Corollary [3, Corollary 3.9]. Let \( f: X \to Y \) be closed and \( X \) regular. If \( f^{-1}(y) \) is closed for every \( y \in Y \), then \( f \) has a closed graph.

Proof. Theorems 3.3 and 3.2.

3.5. Theorem. Let \( f: X \to Y \) be closed and monotone where \( X \) is regular. Then \( C^{-1}(f; y) \) is connected for every \( y \in Y \).

Proof. By Theorem 3.3, \( C^{-1}(f; y) = \text{Cl}(f^{-1}(y)) \) for every \( y \in Y \). Thus, \( C^{-1}(f; y) \) is the closure of the connected set \( f^{-1}(y) \), hence connected.

We have seen that for a given function \( f: X \to Y \), \( C^{-1}(f; y) \) is closed for every \( y \in Y \). The following definition is used to determine a sufficient condition for a union of such sets to remain closed.

3.6. Definition. Let \( f: X \to Y \) and let \( A \subseteq Y \). Then \( C^{-1}(f; A) = \bigcup \{ C^{-1}(f; a) : a \in A \} \).

3.7. Theorem. Let \( f: X \to Y \). If \( A \subseteq Y \) is compact, then \( C^{-1}(f; A) \) is closed.

Proof. First observe that

\[
C^{-1}(f; A) = \bigcup \{ C^{-1}(f; a) : a \in A \} \subseteq \bigcap \{ \text{Cl}(f^{-1}(V)) : V \text{ open and } A \subseteq V \}.
\]

We now show the reverse inclusion. Let \( x \in \bigcap \{ \text{Cl}(f^{-1}(V)) : V \text{ open and } A \subseteq V \} \) and assume that for all \( a \in A \) the filterbase \( f^{-1}(\mathcal{N}(a)) \) does not accumulate to \( x \in X \). Then for each \( a \in A \) there exists a \( V(a) \in \mathcal{N}(a) \) and a \( U_a \in \mathcal{R}(x) \) such that \( f^{-1}(V(a)) \cap U_a = \emptyset \). Now let \( \{ V(a)_i : 1 \leq i \leq n \} \) be a finite subcollection of \( \{ V(a) : a \in A \} \) which covers \( A \) and let \( \{ U_{a(i)} : 1 \leq i \leq n \} \) be the corresponding neighborhoods of \( x \). It follows that

\[
\bigcap \{ U_{a(i)} : 1 \leq i \leq n \} \cap f^{-1}(\bigcup \{ V(a)_i : 1 \leq i \leq n \}) = \emptyset
\]

so that \( x \notin \bigcap \{ C^{-1}(f^{-1}(V)) : V \text{ open and } A \subseteq V \} \). But this contradicts our
hypothesis. We conclude \( x \in C^{-1}(f; A) \) and this implies \( C^{-1}(f; A) = \bigcap \{ \text{Cl}(f^{-1}(V)) : V \text{ open and } A \subseteq V \} \).

The following Corollary again shows how one of Fuller’s results [3] may be obtained using inverse cluster sets.

3.8. Corollary [3, Theorem 3.6]. Let \( f: X \to Y \) be a given function with closed graph. If \( A \subseteq Y \) is compact, then \( f^{-1}(A) \) is closed.

Proof. Theorem 3.2 along with Theorem 3.7 shows that

\[
f^{-1}(A) = \bigcup \{ f^{-1}(a) : a \in A \} = \bigcup \{ C^{-1}(f; a) : a \in A \} = C^{-1}(f; A).
\]

3.9. Theorem. Let \( f: X \to Y \) be continuous from the \( H \)-closed space \( X \) into the Hausdorff space \( Y \). Then \( f \) maps regular-closed sets onto closed sets.

Proof. Let \( M \) be a regular-closed subset of \( X \). It follows that \( M \) is an \( H \)-closed subspace of \( X \). Now consider any \( y \in \text{Cl}(f(M)) \). Since \( f^{-1}(\text{H}(y)) \) is an open filterbase with a trace on \( M \), \( f^{-1}(\text{H}(y)) \) accumulates to some \( x \in M \) [1, Theorem 3.2] so that by Theorem 2.1(3), \( x \in C^{-1}(f; y) \). The fact that \( f \) has a closed graph, along with Theorem 2.2, implies \( x \in C^{-1}(f; y) = f^{-1}(y) \) so that \( y \in f(M) \). We conclude \( f(M) \) is closed.

4. Connectivity functions and inverse cluster sets. For a given function \( f: X \to Y \) where both \( X \) and \( Y \) are first countable, the inverse cluster set \( C^{-1}(f; y) \) is precisely the set \( T(f; y) \) as defined in [5, Definition 3.2]. We now show how cluster sets may be used to generalize Theorem 3.6 of [5] after recalling that a connected function \( f: X \to Y \) is one that preserves connected sets and a connectivity function is one such that the induced function \( g: X \to X \times Y \), defined by \( g(x) = (x, f(x)) \), is connected.

4.1. Theorem. Let \( f: X \to Y \) be a connected function where \( X \) is compact. Then \( f \) is a connectivity function if for each connected \( M \subseteq X \) and any \( x \in M \), \( C^{-1}(f; f(x)) \cap \text{Cl}(M) = \{x\} \).

Proof. Let \( f \) be connected and assume the given condition. Suppose there exists a connected \( M \subseteq X \) such that \( g(M) = H \cup K \) where \( H \) and \( K \) are separated and define \( A = g^{-1}(H) \cap M \) and \( B = g^{-1}(K) \cap M \). Then for any \( x \in A \), there exist open sets \( U \in \text{H}(x) \) and \( V \in \text{H}(f(x)) \) such that \( (U \times V) \cap K = \emptyset \). Consequently, no point of \( U \cap B \) can map into \( V \) under \( f \). Since \( f(M) = f(A) \cup f(B) \) and neither \( f(A) \) nor \( f(B) \) can be empty, we proceed to show \( f(A) \) and \( f(B) \) are separated, thereby obtaining a contradiction. Suppose \( f(x) \in \text{Cl}(f(B)) \) for \( x \in A \). Then \( f^{-1}(\text{H}(y)) \cap B \) is a filterbase on the compact set \( \text{Cl}(M) \) and, hence, accumulates to some \( x_0 \in \text{Cl}(M) \). Therefore, \( x_0 \in C^{-1}(f; f(x)) \), and, since \( f(U \cap B) \cap V = \emptyset \), we have \( \text{Cl}(f^{-1}(V) \cap B) \cap U = \emptyset \). Consequently, \( f(x) \notin C^{-1}(f; f(x)) \cap \text{Cl}(M) \) which contradicts the given
condition of the theorem. We conclude \( f(x) \notin \text{Cl}(f(B)) \) for every \( x \in A \) and, likewise, \( f(x) \notin \text{Cl}(f(A)) \) for every \( x \in B \). This implies \( f(M) \) is not connected. Since \( f \) is given as a connected function, it must follow that \( g \) is connected.

The following Lemma and Theorem give a more workable insight into the condition stated in Theorem 4.1.

4.2. Lemma. Let \( f: X \to Y \) be a given function and let \( A \subset X \). If \( C^{-1}(f(A); y) \) denotes the inverse cluster set of \( f|A: A \to Y \) where \( A \) has the subspace topology, then \( C^{-1}(f(A); y) \subset C^{-1}(f; y) \cap A \) and the equality holds provided \( A \) is open.

Proof. The proof consists of the following set relationships:

\[
C^{-1}(f|A; y) = \bigcap \{ \text{Cl}_A((f|A)^{-1}(V)); V \in \mathcal{N}(y) \}
\]

\[
= \bigcap \{ \text{Cl}_A(f^{-1}(V) \cap A); V \in \mathcal{N}(y) \}
\]

\[
\subset \bigcap \{ \text{Cl}(f^{-1}(V)) \cap A; V \in \mathcal{N}(y) \}
\]

\[
= \bigcap \{ \text{Cl}(f^{-1}(V)); V \in \mathcal{N}(y) \} \cap A
\]

\[
= C^{-1}(f; y) \cap A.
\]

Observe that if \( A \) is open, the subset relation in the proof is an equality.

4.3. Theorem. Let \( f: X \to Y \) be a given function and consider the following conditions:

(1) For each connected set \( M \subset X \) and \( x \in M \), \( C^{-1}(f; f(x)) \cap \text{Cl}(M) = \{x\} \).

(2) For each component \( C \) of \( X \), \( f|C: C \to f(C) \) is a bijection with a closed graph.

Then (1) implies (2) and if \( X \) is locally connected, (1) and (2) are equivalent.

Proof. To show (1) implies (2), let \( C \) be a component of \( X \). Then for each \( x \in C \), where \( y = f(x) \), we have by Lemma 4.2 and the fact that \( C \) is closed,

\[
C^{-1}(f|C; y) \subset C^{-1}(f; y) \cap C = C^{-1}(f; y) \cap \text{Cl}(C) = \{x\}.
\]

Thus, \( f|C \) is a bijection by Theorem 2.11. Since \( C^{-1}(f|C; y) = f^{-1}(y) \), Theorem 3.2 gives the graph of \( f \) closed.

Now assume (2) holds where \( X \) is locally connected, \( M \subset X \) is connected and \( x \in M \). Let \( C \) be the component of \( X \) containing \( \text{Cl}(M) \) and recall that components of locally connected spaces are open so that Lemma 4.2 holds. Then we have

\[
C^{-1}(f|C; y) \cap \text{Cl}(M) \subset C^{-1}(f; f(x)) \cap C = C^{-1}(f|C; f(x)) = \{x\}.
\]

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