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A GENERAL PROOF OF BING'S SHRINKABILITY CRITERION

A. MARIN AND Y. M. VISETTI

ABSTRACT. This paper gives a proof of the general Bing shrinkability criterion, including a proof of the fundamental theorem that a shrinkable compact upper semicontinuous decomposition of a complete metric space is realized by a pseudo-isotopy of the space.

Bing's shrinkability criterion offers a necessary and sufficient condition that a proper[1] surjective mapping $f: X \longrightarrow Y$ of a complete metric space X be strongly approximable by homeomorphisms. The criterion requires roughly that the compacta $f^{-1}(y)$ of the induced decomposition $\{f^{-1}(y) | \; y \in Y\}$ of X should be simultaneously shrinkable to arbitarry small size by self-homeomorphisms of X.

Our aim in this note is to make this criterion more accessible by providing a clear and direct proof of the most general version known, that for complete metric space. It is, we believe, the first such proof that succeeds without engendering complications much greater than one must face in the simplest useful cases.

This shrinkability criterion arose in R. H. Bing's celebrated construction [1] in 1951 of an involution of the 3-sphere S^3 whose fixed point set is a wild ('horned') 2-sphere. It evolved further in his study of the dogbone space X^3 [2, §§1, 7], [3, Theorem 3], a nonmanifold whose product with the line R^1 is homeomorphic to R^4, and was finally formulated for locally compact metric spaces by L. F. McAuley [6], [7]. Then, in 1970, R. D. Edwards and L. C. Glaser [5] gave the version for complete metric spaces and the construction part of a rather complex proof.

We suggest that in applications one should try to verify the complete metrizability of X by establishing either that X is locally compact and metrizable (see Remark 1.3.II below), or by expressing X as a "G_δ" subset of a complete metric space (see [4, p. 308] for justification). Recall that a G_δ set is a countable intersection of open sets; there are many nonlocally compact ones, even in R^n, $n \geq 1$.

In a final section we give elementary proofs of some useful classical

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[1] Proper means that the point preimages $f^{-1}(y)$, $y \in Y$, are compact, and f is closed, i.e. $f(C)$ is closed if C is closed in X.

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results on metric properties that Y inherits from X under a proper surjective map $f\colon X \to Y$.

We suppose only two nontrivial results from general topology; see Dugundji [4]:

(1) Every metric space is paracompact.

(2) The image of a paracompact space under a closed map (e.g. a proper map) is paracompact.

L. C. Siebenmann originally raised the question of a minimal proof for Bing's criterion. We are most grateful to R. D. Edwards for his comments and encouragement. In particular he contributed one of the simplifications in this proof of Bing's criterion.

Generalized Bing's criterion. Let M be a metric space, and $\mathcal{G}$ an upper *semicontinuous* (= u.s.c.) decomposition of M into compact subsets, that is, a collection $\{G \mid G \in \mathcal{G}\}$ of disjoint compacta, whose union is all of M, such that: (u.s.c.) the quotient map $p\colon M \to M/\mathcal{G}$ of M into the decomposition space is closed. $M/\mathcal{G}$ is here the topological quotient space of M by the equivalence relation whose classes are elements G of $\mathcal{G}$.

For convenience of exposition, we initially present a version with isotopies of the Bing criterion. Other variants will be indicated at the end of §1.

We say that such a decomposition is *shrinkable* if, given a saturated open cover $\mathfrak{U}$ of M (saturated means that, for any $G \in \mathcal{G}$ and $U \in \mathfrak{U}$, $G \cap U \neq \emptyset$ implies $G \subset U$), and any open cover $\mathfrak{V}$ of M, there is an isotopy (i.e. homotopy through homeomorphisms) $h_t\colon M \to M$, $t \in [0, 1]$, such that $h_0 = \mathrm{id}_M$ and, for each $G \in \mathcal{G}$:

(i) there is a $U \in \mathfrak{U}$ such that $h_t(G) \subset U$ for all $t \in [0, 1]$, and

(ii) there is a $V \in \mathfrak{V}$ such that $h_1(G) \subset V$.

Theorem 1.1. *Suppose that $\mathcal{G}$ is an u.s.c. and shrinkable decomposition into compact subsets of a complete metric space M. Then, given any saturated open cover $\mathfrak{U}$ of M, there is a pseudo-isotopy $h_t\colon M \to M$, $t \in [0, \infty]$ of $h_0 = \mathrm{id}_M$ (i.e. a homotopy with h_t, $t \in [0, \infty)$ an isotopy), such that:*

(1) *h_∞ factors through $M/\mathcal{G}$ (i.e. $h_\infty = h'_\infty p$) and $h'_\infty\colon M/\mathcal{G} \to M$ is a homeomorphism*

$$\begin{tikzcd} M \arrow[rr, "h_\infty"] \arrow[dr, "p"'] & & M \\ & M/\mathcal{G} \arrow[ur, "h'_\infty"'] & \end{tikzcd}$$

(2) *for each $G \in \mathcal{G}$, there is a $U \in \mathfrak{U}$ such that $h_t(G) \subset U$ for all $t \in [0, \infty]$.*

This h_t is called a $\mathfrak{U}$-pseudo-isotopy collapsing $\mathcal{G}$.

The reader will understand better the meaning of this theorem if he pauses to verify that, conversely, the existence of the pseudo-isotopy h_t for arbitrary $\mathfrak{U}$ implies the shrinkability of $\mathcal{G}$, provided M is compact.

Proof of Theorem 1.1. Let d denote a complete metric on M. If $\epsilon > 0$ and $A \subset M$, let $N_\epsilon(A) = N(A; \epsilon)$ denote $\{x \in M \mid d(x, A) < \epsilon\}$. One fixes a sequence $\epsilon_0, \epsilon_1, \epsilon_2, \cdots$ of positive numbers such that $\epsilon_0 = \infty$ and $\lim \epsilon_n = 0$ as $n \to \infty$. The proof falls naturally into two parts.

Part 1. *The analysis.* We postpone to Part II an easy induction argument, using the shrinkability of $\mathcal{G}$, which allows us to construct two things:

The first is a sequence $\mathfrak{U}_0, \mathfrak{U}_1, \mathfrak{U}_2, \cdots$ of successively finer saturated open covers of M such that the closed cover $\overline{\mathfrak{U}}_0 = \{\overline{U} \mid U \in \mathfrak{U}_0\}$ refines $\mathfrak{U}$, and, for all n, $\mathfrak{U}_n$ refines $\{N(G; \epsilon_n) \mid G \in \mathcal{G}\}$.

The second is an isotopy $h(x, y) = h_t(x)$, $t \in [0, \infty)$, of $h_0 = \mathrm{id}_M$, such that for all $n \geq 0$:

(α_n) For each $G \in \mathcal{G}$, there is a $U \in \mathfrak{U}_n$ such that $h(V \times [n, n+1]) \subset h_n(U)$, for all $V \in \mathfrak{U}_{n+1}$ containing G.

(β_n) For each $U \in \mathfrak{U}_n$, $\operatorname{diam} h_n(U) < \epsilon_n$.

Note that the properties (α_p), $p \geq n$, and (β_n) together imply

(γ_n) *For each $G \in \mathcal{G}$, there is a $U \in \mathfrak{U}_n$ such that $h(G \times [n, \infty)) \subset h_n(U)$ and* $\operatorname{diam} h(G \times [n, \infty)) < \epsilon_n$.

We shall now conclude the proof of Theorem 1.1 using only the properties (α_n), (β_n) and (γ_n).

Claim 1. *By (γ_n), and the completeness of M, the continuous family $h_t: M \to M$, $t \in [0, \infty)$, converges uniformly, as $t \to \infty$, to a continuous map $h_\infty: M \to M$ which factors through $M/\mathcal{G}$ via h'_∞.* (Proof is immediate.)

The following is the one delicate point of the proof:

Claim 2. *h'_∞ (or equivalently h_∞) is onto.*

Proof of Claim 2. We use

Lemma 1.2. *Let M be a metric space, and $x(n)$, $n \in \mathbf{N}$, a sequence of points in M which satisfies the hypothesis*

(H) *For each $\epsilon > 0$, there is a compact G in M with the property that $N_\epsilon(G)$ contains $x(n)$ for all but a finite number of indices n.*

Then there exists a subsequence $x(\phi(n))$ which is Cauchy in M.

Proof of Lemma 1.2. Noting that a subsequence of a sequence which satisfies (H), does also, we can suppose that $\operatorname{diam}\{x(n) \mid n \in \mathbf{N}\} = d < \infty$. Now (H) provides a compact G in M such that, for large n, $x(n) \in N(G; d/6)$. For each such n, choose $z(n)$ in G with $d(x(n), z(n)) < d/6$ and a converging subsequence $z(\phi(n))$, so that $d(z(\phi(n)), z(\phi(m))) < d/6$ for all n, m.

Clearly, $\operatorname{diam}\{x(\phi(n)) \mid n \in \mathbf{N}\} < d/2$. The above procedure allows us to extract successively finer subsequences of $\{x(n)\}$: $\{x(\phi_1(n))\}$, $\{x(\phi_2(n))\}$, $\cdots$, $\{x(\phi_p(n))\}, \ldots$, $p \in \mathbf{N}$, such that $\operatorname{diam}\{x(\phi_p(n)) \mid n \in \mathbf{N}\} < d/2^p$ for all $p > 1$. Then take the diagonal sequence $x(\phi_p(p))$. $\square$

We return to the proof of Claim 2. Recall that $h_\infty = \lim h_n$ as $n \to \infty$, where each h_n is a homeomorphism; let y be in M and x_n be the sequence such that $h_{n+1}(x_n) = y$. Applying (α_n) to the unique $G_n \in \mathcal{G}$ which contains x_n, one gets a saturated $U_n \in \mathfrak{U}_n$ such that $G_n \subset U_n$ and $h_n(U_n) \supset h_n(V) \cup h_{n+1}(V)$ for all $V \in \mathfrak{U}_{n+1}$ containing G_n. In particular $h_n(U_n) \supset h_{n+1}(G_n)$.

Following R. D. Edwards, we observe that the sequence U_n, so constructed is *decreasing*. Indeed, $y = h_{n+2}(x_{n+1}) \in h_{n+2}(G_{n+1}) \subset h_{n+1}(U_{n+1})$, so $x_n = h_{n+1}^{-1}(y) \in U_{n+1}$, whence $G_n \subset G_{n+1}$. This implies, using (α_n), that $h_n(U_n) \supset h_n(U_{n+1})$, so $U_n \supset U_{n+1}$.

As $\mathfrak{U}_n$ refines $\{N(G; \epsilon_n) | \; G \in \mathcal{G}\}$, Edwards' observation implies that the sequence x_n verifies hypothesis (H) of the Lemma. This Lemma provides a limit x for a subsequence $x(\phi(n))$, and, by continuity, $h_\infty(x) = y$.

Claim 3. *h'_∞ is injective and closed.*

Proof of Claim 3. h_∞ is onto, so it suffieces to prove the following. For each saturated closed set F of M, and each $G \in \mathcal{G}$ not in F, there exists an open set Ω in M containing $h_\infty(G)$ and not intersecting $h_\infty(F)$.

Denote $N(F; \mathfrak{U}_n)$ and $N(G; \mathfrak{U}_n)$ the union of all $U \in \mathfrak{U}_n$ which meet F and G respectively. Fix a sufficiently large integer n such that $\overline{N(F; \mathfrak{U}_n)} \cap \overline{N(G; \mathfrak{U}_n)} = \emptyset$. This is possible in view of (β_n) and the fact that $\operatorname{dist}(F, G) > 0$. Now (γ_n) implies $h_\infty(F) \subset h_n\overline{(N(F; \mathfrak{U}_n))}$ and $h_\infty(G) \subset h_n\overline{(N(G; \mathfrak{U}_n))}$. We can choose $\Omega = M - h_n\overline{(N(F; \mathfrak{U}_n))}$.

Part II. *Construction of the covers $\mathfrak{U}_n$ and the isotopy $h_t(x) = h(x, t)$, $t \in [0, \infty)$.*

The induction starts with $h_0 = \mathrm{id}_M$ and $\mathfrak{U}_0$ such that $\overline{\mathfrak{U}}_0 = \{\overline{U} | \; U \in \mathfrak{U}_0\}$ refines $\mathfrak{U}$ and, for each $U \in \mathfrak{U}_0$, $\operatorname{diam} U < \epsilon_0$. For the induction step, given $h(x, t)$, $t \in [0, n]$ and $\mathfrak{U}_n$, consider on M the (shrinkable!) decomposition $h_n\mathcal{G} = \{h_n(G) | \; G \in \mathcal{G}\}$, the saturated open cover $\{h_n(U) | \; U \in \mathfrak{U}_n\}$, and an open cover by sets of diameter smaller than ϵ_{n+1}. Let H_t, $t \in [n, n+1]$, be an associated isotopy provided by the shrinkability. Define $h_t = H_t h_n$ for all $t \in [n, n+1]$.

Now as the elements of $\mathcal{G}$ are compact, we can form a $\mathcal{G}$-saturated open cover $\mathfrak{W}$ which reminds both $\mathfrak{U}_n$ and $\{N(G; \epsilon_{n+1}) | \; G \in \mathcal{G}\}$ and enjoys the following additional properties: for each $W \in \mathfrak{W}$, there exists a $U \in \mathfrak{U}_n$ with $h(W \times [n, n+1]) \subset U$, and $\operatorname{diam} h_{n+1}(W) < \epsilon_{n+1}$. Then take for $\mathfrak{U}_{n+1}$ the pull-back by $p: M \to M/\mathcal{G}$ of an open star refinement of $\{p(W) | \; W \in \mathfrak{W}\}$ provided by *paracompactness* of $M/\mathcal{G}$ [4, p. 168] (in this case, one can obtain such a refinement by a direct argument, but we omit it).

This completes the proof of Theorem 1.1.

1.3. **Remarks on Theorem 1.1.** I. ("localisation"). In many applications all nontrivial (i.e. not a point) compacta of the decomposition lie in a certain open set $U \subset N$ and the shrinkability will be known to hold for the decomposition of U induced by $\mathcal{G}$. Recall that U is complete metrizable for the distance:

$$\delta(x, y) = \sup\{d(x, y), |d(x, M - U)^{-1} - d(y, M - U)^{-1}|\}.$$

For each $G \in \mathcal{G}$; let ϵ_G be $d(G, M - U)/2$ and $\mathfrak{U}$ a saturated open cover of U refining $\{N(G; \epsilon_G)\}$. Any $\mathfrak{U}$-pseudo-isotopy provided by the theorem extends trivially by the identity outside U.

II. Every locally compact metric space N is complete metrizable. This is clear if M is, in addition, sigma compact, since M is then open in its (complete) metrizable one-point compactification. In general, a locally compact metrizable M is at least a disjoint union of open sigma-compact subsets [4, p. 241].

III. With two obvious modifications of the shrinkability hypothesis, one gets two other versions. First, if one works with proper (= closed), isotopies, one gets a proper pseudo-isotopy $h(x, t)$, $t \in [0, \infty]$. Next, one can work without isotopy, so that the theorem just exhibits the quotient map $p: M \longrightarrow M/\mathcal{G}$ as a limit of homeomorphisms. In this case $h(x, t)$, $t \in \{\mathbf{N} \cup \infty\}$ is proper. In both cases, the properness follows quite easily from the following properties (we write A for $\mathbf{N}$ or $[0, \infty)$):

(1) $H: M \times A \longrightarrow M \times A$, $H(x, t) = (h(x, t), t)$ is proper.

(2) h_∞ is a proper surjective map.

(3) $h_t(x)$ converges uniformly to h_∞ as $t \longrightarrow \infty$.

IV. In the case of Bing's theorem without isotopies (in III above), the conclusion is equivalent to the assertion that the quotient map $p: M \longrightarrow M/\mathcal{G}$ be strongly approximable (= majorant approximable) by homeomorphisms, that is, given a cover $\mathfrak{U}$ of $M/\mathcal{G}$, there exists a homeomorphism $h: M \longrightarrow M/\mathcal{G}$ so that: $\forall x \in M$, $\exists U \in \mathfrak{U}$ with $p(x)$ and $h(x)$ in U. Conversely, if $p: M \longrightarrow M/\mathcal{G}$ is strongly approximable by homeomorphisms (with M complete and $\mathcal{G}$ u.s.c.) *then $\mathcal{G}$ is shrinkable* (without isotopies). The proof in either direction uses at most paracompactness in a careful unravelling of the definitions.

2. **Supplementary results.** As a corollary to Theorem 1.1, we obtain that the decomposition space is complete metrizable. This result holds under weaker assumptions:

Theorem 2.1 [8], [9]. *Let M be a metric space and $\mathcal{G}$ an u.s.c. decomposition of M into (merely) closed sets. Then the following are equivalent:*

(i) *$M/\mathcal{G}$ is metrizable.*

(ii) *$M/\mathcal{G}$ is first countable (i.e. each point has a countable neighborhood basis).*

(iii) *Each $G \in \mathcal{G}$ has a compact frontier* $\mathrm{Fr}(G)$ *in M.*

Complement 2.2 [10]. *Suppose M is a complete metric space and the quotient space $M/\mathcal{G}$ by an u.s.c. decomposition is metrizable (that is, one* *of these three equivalent conditions holds*). *Then* $M/\mathcal{G}$ *admits a complete metric.*

For the easy implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), we refer to [8, p. 10] or [9, p. 691].

For (iii) $\Rightarrow$ (i) and the complement, *we can suppose that the closed sets of the decomposition are compact*, as explained in [4, p. 254, Exercise 15]. In this more specific context, (iii) $\Rightarrow$ (i) is nothing else than Theorem 5.2 in [4, p. 235]. We prepare now to prove the complement.

Notations. Denote by d (resp. δ) and compatible metric on M (resp. $M/\mathcal{G}$). A basis of neighborhoods for each (compact) $G \in \mathcal{G}$ is given by $N_\epsilon(G)$ as $\epsilon > 0$ varies. If x is a point in $M/\mathcal{G}$, denote by G the preimage $p^{-1}(x)$.

Definition of a new metric Δ **on** $M/\mathcal{G}$. Let $\epsilon > 0$ be arbitrary; consider on $M/\mathcal{G}$ a locally finite open cover $\mathcal{V} = (V_i)$, $i \in I$, which refines $\{p(N_\epsilon(G)) \mid G \in \mathcal{G}\}$ and $\mathcal{W} = (W_i)$, $i \in I$, a shrinking of $\mathcal{V}$ (that is, $M/\mathcal{G} = \bigcup W_i$ and $\overline{W}_i \subset V_i$). For each i in I, choose a Urysohn function $\psi_i = 1$ on W_i and $\psi_i = 0$ outside V_i, and define a continuous $\delta_\epsilon: M/\mathcal{G} \times M/\mathcal{G} \to [0, 1]$ by $\delta_\epsilon(x, y) = \sup_i |\psi_i(x) - \psi_i(y)|$. Clearly, if the diameter (in the sense of δ_ϵ) of a set $A \subset M/\mathcal{G}$ is smaller than 1, then there exists a $N_\epsilon(G)$ which contains $p^{-1}(A)$. Consider now on $M/\mathcal{G}$ the compatible metric Δ defined by

$$\Delta(x, y) = \delta(x, y) + \sum_{n \geq 1} 2^{-n} \delta_{(1/n)}(x, y).$$

Remark 2.3. A simple direct argument shows that, actually, $\sum_{n \geq 1} 2^{-n}\delta_{(1/n)}(x, y)$ is a compatible metric for $M/\mathcal{G}$. The metrizability of the decomposition space is so deduced from its paracompactness without utilizing any complex criterion.

Proof of Complement 2.2. Let $y(n)$, $n \in N$, be a Cauchy sequence for Δ in $M/\mathcal{G}$. We must find a convergent subsequence for $y(n)$. For each n, choose $x(n)$ in M so that $p(x(n)) = y(n)$.

Fix an integer q. Thus, for large n, m, one has $\Delta(y(n), y(m)) < 2^{-q}$, which implies that $\delta_{(1/q)}(y(n), y(m)) < 1$, so there exists a $N(G; 1/q)$ which contains $\{x(n)\}$ for large n. Lemma 1.2 now provides a convergent subsequence $x(\phi(n))$; as $p: M \to M/\mathcal{G}$ is continuous, $y(\phi(n))$ is also convergent.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PARIS, ORSAY, FRANCE