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CLOSURE-PRESERVING FAMILIES AND METACOMPACTNESS

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ABSTRACT. It is the purpose of this paper to show that if a space X admits a closure-preserving cover of compact, closed sets, then X is metacompact. This paper also provides a characterization of those closure-preserving covers of compact sets admitted by σ-compact spaces.

Introduction. The study of spaces which admit a closure-preserving cover of compact sets was initiated separately by Tamano [5] and Telgarsky [6]. The best results previously available were these [3], [4]: (1) If X is collectionwise-normal and admits a closure-preserving cover consisting of compact sets, then X is metacompact, hence paracompact. (2) If there is an integer n such that X admits a closure-preserving cover of finite sets, each member having no more than n points, then X is metacompact, whether or not it is collectionwise-normal.

In this paper, we show that if a space X admits a closure-preserving cover of compact, closed sets, whether finite or not, then X is metacompact; and this is true without X being collectionwise-normal. We also characterize the closure-preserving covers of compact sets admitted by σ-compact spaces.

When $\mathfrak{L}$ is a family of subsets of some set S, let $|\mathfrak{L}| = \bigcup\{L \mid L \in \mathfrak{L}\}$, and when A is a subset of S, let $(\mathfrak{L})_A = \{L \in \mathfrak{L} \mid L \cap A \neq \varnothing\}$.

The symbol $\mathfrak{F}$ always means a closure-preserving family consisting of compact and closed subsets of a topological space X.

For each x in the space X, $K(x)$ is defined to be $X - \bigcup\{F \in \mathfrak{F} \mid x \notin F\}$.

Lemma 1. *For each $x \in X$, the set $K(x)$ is an open set about x, and when $x \in K(y)$, then $K(x) \subset K(y)$.*

Proof. $\mathfrak{F}$ is closure-preserving, and consists of closed sets, so that $\bigcup\{F \in \mathfrak{F} \mid x \notin F\}$ is closed, and $K(x)$ is seen to be open. Clearly, $x \in K(x)$. To see that $x \in K(y)$ implies $K(x) \subset K(y)$, note that $K(x) = \{p \in X \mid \text{if } p \in F \in \mathfrak{F}, \text{ then } x \in F\}$.

When A is a subset of X, let $M(A)$ be the set $\{x \mid x \in A \cap |\mathfrak{F}| \mid K(x)$ is not properly contained in any set $K(x'), x' \in A\}$.

Lemma 2. (a) *For each $x \in A - M(A)$, the set $K(x) \cap M(A)$ is empty.* (b) *When $x, y \in M(A)$, either $K(x) = K(y)$ or $K(x) \cap K(y) \cap M(A) = \varnothing$.* (c) *If*

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A is closed, $K(x) \cap M(A)$ *is compact for every* $x \in M(A)$.

The proof of this lemma is essentially that of Lemma 3 of [3].

Lemma 3. *When* A *is a subset of* X, *let* $\mathcal{K}(A)$ *be the family* $\{K(x) \mid x \in M(A)\}$. *Let* $\mathcal{F}$ *be a closure-preserving family of compact closed subsets of the space* X, *and let* A *be a closed subset of* X. *Then the family* $\mathcal{K}(A)$ *covers* $A \cap |\mathcal{F}|$ *and, for each* $F \in \mathcal{F}$, *the family* $(\mathcal{K}(A))_F$ *is finite.*

Proof. To show that the family $\mathcal{K}(A)$ covers the set $A \cap |\mathcal{F}|$, let F be a member of the family $\mathcal{F}$. The closed subset $A \cap F$ of the compact set F is compact so that there is a finite subset B of $A \cap F$ such that $A \cap F \subset \bigcup\{K(b) \mid b \in B\}$. Since B is finite, we have $A \cap F \subset |\mathcal{K}(B)|$. We now show that $M(B) \subset M(A)$. Let b be a point of $M(B)$ and c a point of A such that $K(b) \subset K(c)$. Then $c \in F$, else $K(c) \cap F$ would be empty and $K(b) \subset K(c)$ would be impossible. Hence the point c is in $A \cap F$ and so there is a point $b' \in M(B)$ such that $c \in K(b')$. By Lemma 1, we have $K(c) \subset K(b')$ and hence $K(b) \subset K(b')$ as well. Since b, b' are both in $M(b)$, we must have $K(b) = K(b')$. But then we have $K(b) \subset K(c) \subset K(b')$, and $K(b) = K(b')$, hence $K(c) = K(b)$. We have shown that for no $c \in A$ is the set $K(b)$ properly contained in the set $K(c)$ and, since b is in $A \cap |\mathcal{F}|$, $b \in M(A)$. Hence it follows that $M(B) \subset M(A)$. Since we have $M(B) \subset M(A)$ and $A \cap F \subset |\mathcal{K}(B)|$, we have that $A \cap F$ is covered by the family $\mathcal{K}(A)$. We now have that all sets $A \cap F$, where $F \in \mathcal{F}$, are covered by the family $\mathcal{K}(A)$, whence it follows that $A \cap |\mathcal{F}|$ is covered by the family $\mathcal{K}(A)$.

To show that for each $F \in \mathcal{F}$, the family $(\mathcal{K}(A))_F$ is finite, let F be a member of $\mathcal{F}$. From what has been shown above, it follows that there is a finite subfamily $\mathcal{K}$ of $\mathcal{K}(A)$ such that $A \cap F \subset |\mathcal{K}|$. We now show that $(\mathcal{K}(A))_F \subset \mathcal{K}$. Let K be a member of the family $(\mathcal{K}(A))_F$. There is a point $x \in M(A)$ such that $K = K(x)$. But then $x \in F$, else we would have $K(x) \cap F = \emptyset$, which contradicts the assumption $K \in (\mathcal{K}(A))_F$. Hence x is in $A \cap F$, and since $A \cap F \subset |\mathcal{K}|$, there is a set $L \in \mathcal{K}$ such that $x \in L$. Let y be a point of $M(A)$ such that $L = K(y)$. Then we have $x \in K(x) \cap K(y) \cap M(A)$ and, from Lemma 2, we can say $K(x) = K(y)$, that is, $K = L$. Hence the set K is a member of the family $\mathcal{K}$, and we have shown that the family $(\mathcal{K}(A))_F$ is contained in the finite family $\mathcal{K}$.

Lemma 4. *Let* $\mathcal{F}$ *be a closure-preserving family of compact, closed subsets of the space* X, *and let* $\{O(n) \mid n \in Z^+\}$ *be a family of open subsets of* X *such that* $M(X) \subset O(1)$, *and for each* $n \in Z^+$, $M(X - \bigcup\{O(i) \mid i = 1, 2, \ldots, n\}) \subset O(n+1)$. *Then* $|\mathcal{F}| \subset \bigcup\{O(n) \mid n \in Z^+\}$.

Proof. Since $\bigcup_{n=1}^{\infty} M(X - \bigcup\{O(i) \mid i = 1, 2, \ldots, n\})$ is a subset of $\bigcup\{O(n) \mid n \in Z^+\}$, we have, by Lemma 2, that for $x \in X - \bigcup\{O(n) \mid n \in Z^+\}$, the set $K(x)$ never intersects the set $\bigcup_{n=1}^{\infty} M(X - \bigcup\{O(i) | i = 1, 2, \ldots, n\})$. Hence, if we set V to be the open set $\bigcup\{K(x) | x \in X - \bigcup_{n=1}^{\infty} O(n)\}$, we have

$$V \cap \left[\bigcup_{n=1}^{\infty} M(X - \bigcup\{O(i) \mid i = 1, 2, \ldots, n\})\right] = \varnothing.$$

Let F be a member of the family $\mathfrak{F}$. We show that $F \subset \bigcup\{O(n) | n \in Z^+\}$. The family $\{V\} \cup \{O(n) | n \in Z^+\}$ is an open cover of the compact set F, hence there is a positive integer n_0 such that $F \subset V \cup (\bigcup\{O(n) | n = 1, 2, \ldots, n_0\})$. We want to show that $F \subset \bigcup\{O(n) | n = 1, 2, \ldots, n_0\}$. Suppose instead that $F - \bigcup\{O(n) | n = 1, 2, \ldots, n_0\}$ is nonempty, and let x be a point of this set. By Lemma 3, there is a point $y \in M(X - \bigcup\{O(n) | n = 1, 2, \ldots, n_0\})$ such that $x \in K(y)$. We must have $y \in F$, else $K(y) \cap F$ would be empty. Since $V \cap M(X - \bigcup\{O(n) | n = 1, 2, \ldots, n_0\}) = \varnothing$, the point y is in the set $F - V$. Since F is contained in $V \cup (\bigcup\{O(n) | n = 1, 2, \ldots, n_0\})$, we see that y is in the set $\bigcup(O(n) | n = 1, 2, \ldots, n_0\}$. But this is a contradiction, since $y \in M(X - \bigcup\{O(n) | n = 1, 2, \ldots, n_0\})$. So the set $F - \bigcup\{O(n) | n = 1, 2, \ldots, n_0\}$ must be empty and F is seen to be covered by the family $\{O(n) | n \in Z^+\}$, and we have $|\mathfrak{F}| \subset \bigcup\{O(n) | n \in Z^+\}$.

Theorem 1. *Let $\mathfrak{F}$ be a closure-preserving family of compact closed subsets of the space X, and let $\mathfrak{U}$ be an open cover of X. Then $\mathfrak{U}$ has an open refinement $\mathfrak{V}$ such that for every $F \in \mathfrak{F}$, the family $(\mathfrak{V})_F$ is finite.*

Proof. We first construct a sequence of open families $\{\mathcal{O}(n) | n \in Z^+\}$ satisfying the following properties:

(1) $\mathcal{O}(0) = \{\varnothing\}$.

(2) $M(X - \bigcup\{\mathcal{O}(i) | i = 0, 1, \ldots, n-1\}) \subset |\mathcal{O}(n)|$.

(3) For each $F \in \mathfrak{F}$, the family $(\mathcal{O}(n))_F$ is finite.

(4) Each set $O \in \mathcal{O}(n)$ is contained in some set $U \in \mathfrak{U}$.

Let $\mathcal{O}(0)$ be $\{\varnothing\}$ and let k be a natural number such that the open families $\mathcal{O}(i)$, $i = 0, 1, \ldots, k-1$ have already been constructed. We show that there is an open family $\mathcal{O}(k)$ satisfying the stated conditions. Let A be the closed set $X - \bigcup\{|\mathcal{O}(i)| i = 0, 1, \ldots, k-1\}$. By Lemma 2, we have that for each $K \in \mathfrak{K}(A)$ there exists a finite subfamily $\mathfrak{U}(K)$ of the open cover $\mathfrak{U}$ such that the family $\mathfrak{U}(K)$ covers the set $K \cap M(A)$. For every $K \in \mathfrak{K}(A)$, let $\mathcal{O}(K)$ be the family $\{U \cap K | U \in \mathfrak{U}(K)\}$ and let $\mathcal{O}(k)$ be the family $\bigcup\{\mathcal{O}(K) | K \in \mathfrak{K}(A)\}$. Then $\mathcal{O}(k)$ is an open family and every one of its sets is contained in some set of the cover $\mathfrak{U}$. It remains to show that conditions (2) and (3) are satisfied for $n = k$. To show that (2) is satisfied, let x be a point of the set $M(X - \bigcup\{|\mathcal{O}(i)| \,|i = 0, 1, \ldots, k-1\})$, in other words, a point of the set $M(A)$. Then the set $K(x)$ is in the family $\mathfrak{K}(A)$ and, thus, there is a set $U \in \mathfrak{U}(K(x))$ such that $x \in U$. The point x is in the set $U \cap K$, which belongs to the subfamily $\mathcal{O}(K(x))$ of the family $\mathcal{O}(k)$, where $x \in |\mathcal{O}(k)|$ and (2) is seen to be satisfied for $n = k$. To show that (3) is satisfied, let F be a member of $\mathcal{F}$. Then by Lemma 3, the family $(\mathcal{K}(A))_F$ is finite. Since $|\mathcal{O}(K)| \subset K$ for every $K \in \mathcal{K}(A)$, we see that the family $(\mathcal{O}(k))_F$ is contained in the family $\bigcup\{\mathcal{O}(K)| K \in (\mathcal{K}(A))_F\}$. But this latter family is a finite union of finite families, so $(\mathcal{O}(k))_F$ is finite, and (3) is satisfied for $n = k$.

To complete the proof of the theorem, first note that when we set $|\mathcal{O}(n)| = O(n)$, the family $\{O(n)|n \in Z^+\}$ satisfies the conditions of Lemma 4, hence we have by that lemma that $|\mathcal{F}| \subset \bigcup\{O(n)|n \in Z^+\}$; in other words $|\mathcal{F}| \subset \bigcup\{|\mathcal{O}(n)|\,|n \in Z^+\}$. Now for each positive integer n, let $L(n) = \bigcup\{F \in \mathcal{F}|F \subset \bigcup_{i=0}^{n-1} \mathcal{O}(i)\}$, and let $\mathcal{O}'(n) = \{O - L(n)|O \in \mathcal{O}(n)\}$. Since the family $\mathcal{F}$ is closure-preserving and consists of closed sets, we have that the sets $L(n)$ are closed and so the families $\mathcal{O}'(n)$ are open. Let $\mathcal{O} = \bigcup\{\mathcal{O}'(n)|n \in Z^+\}$. Since the families $\mathcal{O}(n)$ satisfy (4), we see that every set of the open family $\mathcal{O}$ is contained in some set of the cover $\mathcal{U}$. We show that the family $\mathcal{O}$ covers the set $|\mathcal{F}|$. Let x be a point of the set $|\mathcal{F}|$. Let $n(x)$ be the smallest integer n such that $x \in |\mathcal{O}(n)|$. Such integers exist because $|\mathcal{F}| \subset \bigcup\{|\mathcal{O}(n)|\,|n \in Z^+\}$. Now for $x \in F \in \mathcal{F}$, we have that F is not a subset of $\bigcup\{|\mathcal{O}(n)|\,|n = 0, 1, \ldots, n(x) - 1\}$, so that x does not belong to $L(n(x))$. Moreover, since $|\mathcal{O}'(n(x))| = |\mathcal{O}(n(x))| - L(n(x))$, it follows that x is in the set $|\mathcal{O}'(n(x))|$ and, hence, also in the set $|\mathcal{O}|$, whence the family $\mathcal{O}$ is seen to cover $|\mathcal{F}|$. Next we show that for every $F \in \mathcal{F}$, the family $(\mathcal{O})_F$ is finite. Let F be a member of the family $\mathcal{F}$. Now F is compact and is contained in the union of the open sets $|\mathcal{O}(n)|$, $n \in Z^+$, so that the set $N = \{n \in Z^+|F \subset \bigcup_{k=1}^{n} |\mathcal{O}(k)|\}$ is not empty. Let n_0 be the least member of N. Then we have $F \subset L(n)$ for every $n > n_0$, and it follows then that the family $(\mathcal{O}'(n))_F$ is empty for every $n > n_0$. As the families $\mathcal{O}(n)$ satisfy (3), we see that the family $(\mathcal{O}'(n))_F$ is finite for every $n \in Z^+$. Thus the family $(\mathcal{O})_F$, which can be represented as $\bigcup\{(\mathcal{O}'(n))_F| n \in Z^+\}$, is finite.

Since the set $|\mathcal{F}|$ is closed, it follows that when we set $\mathcal{U}' = \{U - |\mathcal{F}|\,|U \in \mathcal{U}\}$ and $\mathcal{V} = \mathcal{O} \cup \mathcal{U}'$, the family $\mathcal{V}$ has the properties required by the theorem.

There are a number of corollaries available.

Corollary 1. *If a space X has a closure-preserving cover consisting of compact closed sets, then X is metacompact.*

There is an example in [2] of a completely regular, Hausdorff space, which has a closure-preserving cover consisting of finite sets, and hence is metacompact, yet which is not paracompact, nor even normal. However, by including an additional condition on the cover, we obtain a characterization of the class of locally compact paracompact spaces.

Corollary 2. *If a space X has a closure-preserving cover consisting of*

compact closed sets whose interiors cover X, then X is paracompact. In *fact, the existence of such a cover is characteristic of those spaces which are both locally compact and paracompact.*

It is interesting to note that these are paracompact spaces which do not admit even a σ-closure-preserving cover of compact sets. In [3] it is shown that the space of irrationals is such a space. The existence of a σ-closure-preserving cover implies another property, however.

Corollary 3. *If a space X has a cover which can be decomposed into countably many families, each of which is closure-preserving and consists of compact closed sets, then X is θ-refinable.*

Now we proceed to characterize certain types of spaces and covers. If X is a topological space and $\mathfrak{F}$ is a closure-preserving cover consisting of compact closed sets, then we introduce a partial ordering on X by agreeing that $x \leq y$ provided $K(x) \supset K(y)$.

Theorem 2. *If X is compact or σ-compact, then $\mathfrak{F}$ admits a closure-preserving cover of compact, closed sets with respect to which the ordering $\leq$ is a linear ordering.*

Proof. If X is compact, let $\mathfrak{F} = \{X\}$. Then for any points $x, y \in X$, we have $K(x) = K(y) = X$, and so it is certainly true that either $x \leq y$ or $y \leq x$. If X is a σ-compact space, let $\{G(i) | i \in Z^+\}$ be a sequence of compact closed sets whose union is X. For each positive integer i, set $F(i) = \bigcup\{G(j) | j = 1, 2, \ldots, i\}$. Then each set $F(i)$ is a compact, closed set, and $\mathfrak{F} = \{F(i) | i \in Z^+\}$ is a cover of X consisting of compact closed sets. Moreover, it is closure-preserving, for the union of any infinite subcollection of $\mathfrak{F}$ is the entire space X, and the union of any finite number of members of $\mathfrak{F}$, say $F(i_1) \cup F(i_2) \cup \cdots \cup F(i_n)$ is simply that set with the largest index from among $i_1, i_2, \ldots, i_n$. In either case, we have a closed set. In addition, if $\leq$ is defined as above, it is a linear order. Clearly, it is sufficient to prove that if $x, y \in X$, then $K(x) \subset K(y)$, or else $K(y) \subset K(x)$. Let j be the least of the integers $\{i | x \in F(i)\}$; let k be the least of the integers $\{i | y \in F(i)\}$, and assume that $j \leq k$. Then

$$K(x) = X - \bigcup\{F \in \mathfrak{F} \mid x \notin F\} = X - \bigcup\{F(i) \mid i < j\}$$

and

$$K(y) = X - \bigcup\{F \in \mathfrak{F} \mid y \notin F\} = X - \bigcup\{F(i) \mid i < K\}$$

so that $X - \bigcup\{F(i) | i < j\} \supset X - \bigcup\{F(i) | i < k\}$, that is, $K(x) \supset K(y)$, so that $x \leq y$, and $\leq$ is seen to be a linear ordering.

The converse of the previous theorem is also true and we show this via a sequence of lemmas.

Lemma 5. *Let X be a topological space and let $\mathfrak{F}$ be a closure-preserving cover consisting of compact, closed sets. If $\leq$ is a linear ordering and* *there exists a sequence in X which is cofinal with respect to $\leq$, then X is σ-compact.*

Proof. Suppose that $\leq$ is a linear ordering, that is, for $x, y \in X$, either $K(x) \supset K(y)$, or $K(x) \subset K(y)$. Let $Y = \{y(i) | i \in Z^+\}$ be a cofinal sequence with respect to $\leq$. For each positive integer i, choose $F(i) \in \mathfrak{F}$ such that $y(i) \in F(i)$. Since Y is cofinal with respect to $\leq$, for each $x \in X$, there is an integer i such that $x \leq y(i)$, that is, $K(x) \supset K(y(i))$. But then $y(i) \in F(i)$ and $K(x) \supset K(y(i))$ together imply that $x \in F(i)$, so we have that $X = \bigcup\{F(i) | i \in Z^+\}$, and X is seen to be σ-compact.

Lemma 6. *Let X have the same properties as in Lemma 5. Then any infinite sequence in X is either cofinal with respect to $\leq$ or has a limit point.*

Proof. Let $Y = \{y(i) | i \in Z^+\}$ be an infinite sequence. If Y fails to be cofinal with respect to $\leq$, there is an $x \in X$ such that for each positive integer i, we have $y(i) \leq x$, that is $K(y(i)) \supset K(x)$. Choose $F \in \mathfrak{F}$ such that $x \in F$. Then $K(y(i)) \supset K(x)$ and $x \in F$ together imply that $y(i) \in F$. But then Y is an infinite subset of the compact set F, hence must have a limit point.

Lemma 7. *If X is a topological space satisfying the hypotheses of Lemmas 5 and 6, then X is either σ-compact or countably compact.*

Proof. If X has even one cofinal sequence $Y = \{y(i) | i \in Z^+\}$, then by Lemma 5, X is σ-compact. So we suppose that no sequence is cofinal. But then Lemma 6 guarantees that each sequence has a limit point, whence X is countably compact.

Theorem 3. *Let X be a topological space and let $\mathfrak{F}$ be a closure-preserving cover consisting of compact, closed sets. Suppose that $\leq$ is a linear ordering, that is, for $x, y \in X$, either $K(x) \supset K(y)$, or $K(x) \subset K(y)$. Then X is either compact or σ-compact.*

Proof. By Lemma 7, X is either σ-compact or countably compact. By Corollary 1, X is metacompact. But it is well known that countable compactness and metacompactness together imply compactness, so X is either compact or σ-compact.

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