

CLOSURE-PRESERVING FAMILIES AND METACOMPACTNESS

HENRY POTOCZNY AND HEIKKI JUNNILA

ABSTRACT. It is the purpose of this paper to show that if a space X admits a closure-preserving cover of compact, closed sets, then X is metacompact. This paper also provides a characterization of those closure-preserving covers of compact sets admitted by σ -compact spaces.

Introduction. The study of spaces which admit a closure-preserving cover of compact sets was initiated separately by Tamano [5] and Telgarsky [6]. The best results previously available were these [3], [4]: (1) If X is collectionwise-normal and admits a closure-preserving cover consisting of compact sets, then X is metacompact, hence paracompact. (2) If there is an integer n such that X admits a closure-preserving cover of finite sets, each member having no more than n points, then X is metacompact, whether or not it is collectionwise-normal.

In this paper, we show that if a space X admits a closure-preserving cover of compact, closed sets, whether finite or not, then X is metacompact; and this is true without X being collectionwise-normal. We also characterize the closure-preserving covers of compact sets admitted by σ -compact spaces.

When \mathcal{L} is a family of subsets of some set S , let $|\mathcal{L}| = \bigcup\{L \mid L \in \mathcal{L}\}$, and when A is a subset of S , let $(\mathcal{L})_A = \{L \in \mathcal{L} \mid L \cap A \neq \emptyset\}$.

The symbol \mathcal{F} always means a closure-preserving family consisting of compact and closed subsets of a topological space X .

For each x in the space X , $K(x)$ is defined to be $X - \bigcup\{F \in \mathcal{F} \mid x \notin F\}$.

Lemma 1. *For each $x \in X$, the set $K(x)$ is an open set about x , and when $x \in K(y)$, then $K(x) \subset K(y)$.*

Proof. \mathcal{F} is closure-preserving, and consists of closed sets, so that $\bigcup\{F \in \mathcal{F} \mid x \notin F\}$ is closed, and $K(x)$ is seen to be open. Clearly, $x \in K(x)$. To see that $x \in K(y)$ implies $K(x) \subset K(y)$, note that $K(x) = \{p \in X \mid \text{if } p \in F \in \mathcal{F}, \text{ then } x \in F\}$.

When A is a subset of X , let $M(A)$ be the set $\{x \mid x \in A \cap \bigcap\{F \in \mathcal{F} \mid K(x) \text{ is not properly contained in any set } K(x'), x' \in A\}\}$.

Lemma 2. (a) *For each $x \in A - M(A)$, the set $K(x) \cap M(A)$ is empty.* (b) *When $x, y \in M(A)$, either $K(x) = K(y)$ or $K(x) \cap K(y) \cap M(A) = \emptyset$.* (c) *If*

Received by the editors October 31, 1974.

AMS (MOS) subject classifications (1970). Primary 54D20; Secondary 54D15, 54D30.

A is closed, $K(x) \cap M(A)$ is compact for every $x \in M(A)$.

The proof of this lemma is essentially that of Lemma 3 of [3].

Lemma 3. *When A is a subset of X , let $\mathcal{K}(A)$ be the family $\{K(x) \mid x \in M(A)\}$. Let \mathcal{F} be a closure-preserving family of compact closed subsets of the space X , and let A be a closed subset of X . Then the family $\mathcal{K}(A)$ covers $A \cap |\mathcal{F}|$ and, for each $F \in \mathcal{F}$, the family $(\mathcal{K}(A))_F$ is finite.*

Proof. To show that the family $\mathcal{K}(A)$ covers the set $A \cap |\mathcal{F}|$, let F be a member of the family \mathcal{F} . The closed subset $A \cap F$ of the compact set F is compact so that there is a finite subset B of $A \cap F$ such that $A \cap F \subset \bigcup \{K(b) \mid b \in B\}$. Since B is finite, we have $A \cap F \subset |\mathcal{K}(B)|$. We now show that $M(B) \subset M(A)$. Let b be a point of $M(B)$ and c a point of A such that $K(b) \subset K(c)$. Then $c \in F$, else $K(c) \cap F$ would be empty and $K(b) \subset K(c)$ would be impossible. Hence the point c is in $A \cap F$ and so there is a point $b' \in M(B)$ such that $c \in K(b')$. By Lemma 1, we have $K(c) \subset K(b')$ and hence $K(b) \subset K(b')$ as well. Since b, b' are both in $M(b)$, we must have $K(b) = K(b')$. But then we have $K(b) \subset K(c) \subset K(b')$, and $K(b) = K(b')$, hence $K(c) = K(b)$. We have shown that for no $c \in A$ is the set $K(b)$ properly contained in the set $K(c)$ and, since b is in $A \cap |\mathcal{F}|$, $b \in M(A)$. Hence it follows that $M(B) \subset M(A)$. Since we have $M(B) \subset M(A)$ and $A \cap F \subset |\mathcal{K}(B)|$, we have that $A \cap F$ is covered by the family $\mathcal{K}(A)$. We now have that all sets $A \cap F$, where $F \in \mathcal{F}$, are covered by the family $\mathcal{K}(A)$, whence it follows that $A \cap |\mathcal{F}|$ is covered by the family $\mathcal{K}(A)$.

To show that for each $F \in \mathcal{F}$, the family $(\mathcal{K}(A))_F$ is finite, let F be a member of \mathcal{F} . From what has been shown above, it follows that there is a finite subfamily \mathcal{K} of $\mathcal{K}(A)$ such that $A \cap F \subset |\mathcal{K}|$. We now show that $(\mathcal{K}(A))_F \subset \mathcal{K}$. Let K be a member of the family $(\mathcal{K}(A))_F$. There is a point $x \in M(A)$ such that $K = K(x)$. But then $x \in F$, else we would have $K(x) \cap F = \emptyset$, which contradicts the assumption $K \in (\mathcal{K}(A))_F$. Hence x is in $A \cap F$, and since $A \cap F \subset |\mathcal{K}|$, there is a set $L \in \mathcal{K}$ such that $x \in L$. Let y be a point of $M(A)$ such that $L = K(y)$. Then we have $x \in K(x) \cap K(y) \cap M(A)$ and, from Lemma 2, we can say $K(x) = K(y)$, that is, $K = L$. Hence the set K is a member of the family \mathcal{K} , and we have shown that the family $(\mathcal{K}(A))_F$ is contained in the finite family \mathcal{K} .

Lemma 4. *Let \mathcal{F} be a closure-preserving family of compact, closed subsets of the space X , and let $\{O(n) \mid n \in \mathbb{Z}^+\}$ be a family of open subsets of X such that $M(X) \subset O(1)$, and for each $n \in \mathbb{Z}^+$, $M(X - \bigcup \{O(i) \mid i = 1, 2, \dots, n\}) \subset O(n+1)$. Then $|\mathcal{F}| \subset \bigcup \{O(n) \mid n \in \mathbb{Z}^+\}$.*

Proof. Since $\bigcup_{n=1}^{\infty} M(X - \bigcup \{O(i) \mid i = 1, 2, \dots, n\})$ is a subset of $\bigcup \{O(n) \mid n \in \mathbb{Z}^+\}$, we have, by Lemma 2, that for $x \in X - \bigcup \{O(n) \mid n \in \mathbb{Z}^+\}$, the

set $K(x)$ never intersects the set $\bigcup_{n=1}^{\infty} M(X - \bigcup\{O(i) \mid i = 1, 2, \dots, n\})$. Hence, if we set V to be the open set $\bigcup\{K(x) \mid x \in X - \bigcup_{n=1}^{\infty} O(n)\}$, we have

$$V \cap \left[\bigcup_{n=1}^{\infty} M(X - \bigcup\{O(i) \mid i = 1, 2, \dots, n\}) \right] = \emptyset.$$

Let F be a member of the family \mathcal{F} . We show that $F \subset \bigcup\{O(n) \mid n \in \mathbb{Z}^+\}$. The family $\{V\} \cup \{O(n) \mid n \in \mathbb{Z}^+\}$ is an open cover of the compact set F , hence there is a positive integer n_0 such that $F \subset V \cup (\bigcup\{O(n) \mid n = 1, 2, \dots, n_0\})$. We want to show that $F \subset \bigcup\{O(n) \mid n = 1, 2, \dots, n_0\}$. Suppose instead that $F - \bigcup\{O(n) \mid n = 1, 2, \dots, n_0\}$ is nonempty, and let x be a point of this set. By Lemma 3, there is a point $y \in M(X - \bigcup\{O(n) \mid n = 1, 2, \dots, n_0\})$ such that $x \in K(y)$. We must have $y \in F$, else $K(y) \cap F$ would be empty. Since $V \cap M(X - \bigcup\{O(n) \mid n = 1, 2, \dots, n_0\}) = \emptyset$, the point y is in the set $F - V$. Since F is contained in $V \cup (\bigcup\{O(n) \mid n = 1, 2, \dots, n_0\})$, we see that y is in the set $\bigcup\{O(n) \mid n = 1, 2, \dots, n_0\}$. But this is a contradiction, since $y \in M(X - \bigcup\{O(n) \mid n = 1, 2, \dots, n_0\})$. So the set $F - \bigcup\{O(n) \mid n = 1, 2, \dots, n_0\}$ must be empty and F is seen to be covered by the family $\{O(n) \mid n \in \mathbb{Z}^+\}$, and we have $|\mathcal{F}| \subset \bigcup\{O(n) \mid n \in \mathbb{Z}^+\}$.

Theorem 1. *Let \mathcal{F} be a closure-preserving family of compact closed subsets of the space X , and let \mathcal{U} be an open cover of X . Then \mathcal{U} has an open refinement \mathcal{O} such that for every $F \in \mathcal{F}$, the family $(\mathcal{O})_F$ is finite.*

Proof. We first construct a sequence of open families $\{\mathcal{O}(n) \mid n \in \mathbb{Z}^+\}$ satisfying the following properties:

- (1) $\mathcal{O}(0) = \{\emptyset\}$.
- (2) $M(X - \bigcup\{\mathcal{O}(i) \mid i = 0, 1, \dots, n - 1\}) \subset |\mathcal{O}(n)|$.
- (3) For each $F \in \mathcal{F}$, the family $(\mathcal{O}(n))_F$ is finite.
- (4) Each set $O \in \mathcal{O}(n)$ is contained in some set $U \in \mathcal{U}$.

Let $\mathcal{O}(0)$ be $\{\emptyset\}$ and let k be a natural number such that the open families $\mathcal{O}(i)$, $i = 0, 1, \dots, k - 1$ have already been constructed. We show that there is an open family $\mathcal{O}(k)$ satisfying the stated conditions. Let A be the closed set $X - \bigcup\{|\mathcal{O}(i)| \mid i = 0, 1, \dots, k - 1\}$. By Lemma 2, we have that for each $K \in \mathcal{K}(A)$ there exists a finite subfamily $\mathcal{U}(K)$ of the open cover \mathcal{U} such that the family $\mathcal{U}(K)$ covers the set $K \cap M(A)$. For every $K \in \mathcal{K}(A)$, let $\mathcal{O}(K)$ be the family $\{U \cap K \mid U \in \mathcal{U}(K)\}$ and let $\mathcal{O}(k)$ be the family $\bigcup\{\mathcal{O}(K) \mid K \in \mathcal{K}(A)\}$. Then $\mathcal{O}(k)$ is an open family and every one of its sets is contained in some set of the cover \mathcal{U} . It remains to show that conditions (2) and (3) are satisfied for $n = k$. To show that (2) is satisfied, let x be a point of the set $M(X - \bigcup\{|\mathcal{O}(i)| \mid i = 0, 1, \dots, k - 1\})$, in other words, a point of the set $M(A)$. Then the set $K(x)$ is in the family $\mathcal{K}(A)$ and, thus, there is a set $U \in \mathcal{U}(K(x))$ such that $x \in U$. The point x is in the set $U \cap K$, which belongs to the subfamily $\mathcal{O}(K(x))$ of the family $\mathcal{O}(k)$, where $x \in |\mathcal{O}(k)|$ and (2) is seen to be

satisfied for $n = k$. To show that (3) is satisfied, let F be a member of \mathcal{F} . Then by Lemma 3, the family $(K(A))_F$ is finite. Since $|\mathcal{O}(K)| \subset K$ for every $K \in \mathcal{K}(A)$, we see that the family $(\mathcal{O}(k))_F$ is contained in the family $\bigcup\{|\mathcal{O}(K)| \mid K \in \mathcal{K}(A)_F\}$. But this latter family is a finite union of finite families, so $(\mathcal{O}(k))_F$ is finite, and (3) is satisfied for $n = k$.

To complete the proof of the theorem, first note that when we set $|\mathcal{O}(n)| = O(n)$, the family $\{O(n) \mid n \in \mathbb{Z}^+\}$ satisfies the conditions of Lemma 4, hence we have by that lemma that $|\mathcal{F}| \subset \bigcup\{O(n) \mid n \in \mathbb{Z}^+\}$; in other words $|\mathcal{F}| \subset \bigcup\{|\mathcal{O}(n)| \mid n \in \mathbb{Z}^+\}$. Now for each positive integer n , let $L(n) = \bigcup\{F \in \mathcal{F} \mid F \subset \bigcup_{i=0}^{n-1} \mathcal{O}(i)\}$, and let $\mathcal{O}'(n) = \{O - L(n) \mid O \in \mathcal{O}(n)\}$. Since the family \mathcal{F} is closure-preserving and consists of closed sets, we have that the sets $L(n)$ are closed and so the families $\mathcal{O}'(n)$ are open. Let $\mathcal{O} = \bigcup\{\mathcal{O}'(n) \mid n \in \mathbb{Z}^+\}$. Since the families $\mathcal{O}(n)$ satisfy (4), we see that every set of the open family \mathcal{O} is contained in some set of the cover \mathcal{U} . We show that the family \mathcal{O} covers the set $|\mathcal{F}|$. Let x be a point of the set $|\mathcal{F}|$. Let $n(x)$ be the smallest integer n such that $x \in |\mathcal{O}(n)|$. Such integers exist because $|\mathcal{F}| \subset \bigcup\{|\mathcal{O}(n)| \mid n \in \mathbb{Z}^+\}$. Now for $x \in F \in \mathcal{F}$, we have that F is not a subset of $\bigcup\{|\mathcal{O}(n)| \mid n = 0, 1, \dots, n(x) - 1\}$, so that x does not belong to $L(n(x))$. Moreover, since $|\mathcal{O}'(n(x))| = |\mathcal{O}(n(x))| - L(n(x))$, it follows that x is in the set $|\mathcal{O}'(n(x))|$ and, hence, also in the set $|\mathcal{O}|$, whence the family \mathcal{O} is seen to cover $|\mathcal{F}|$. Next we show that for every $F \in \mathcal{F}$, the family $(\mathcal{O})_F$ is finite. Let F be a member of the family \mathcal{F} . Now F is compact and is contained in the union of the open sets $|\mathcal{O}(n)|$, $n \in \mathbb{Z}^+$, so that the set $N = \{n \in \mathbb{Z}^+ \mid F \subset \bigcup_{k=1}^n |\mathcal{O}(k)|\}$ is not empty. Let n_0 be the least member of N . Then we have $F \subset L(n)$ for every $n > n_0$, and it follows then that the family $(\mathcal{O}'(n))_F$ is empty for every $n > n_0$. As the families $\mathcal{O}(n)$ satisfy (3), we see that the family $(\mathcal{O}'(n))_F$ is finite for every $n \in \mathbb{Z}^+$. Thus the family $(\mathcal{O})_F$, which can be represented as $\bigcup\{(\mathcal{O}'(n))_F \mid n \in \mathbb{Z}^+\}$, is finite.

Since the set $|\mathcal{F}|$ is closed, it follows that when we set $\mathcal{U}' = \{U - |\mathcal{F}| \mid U \in \mathcal{U}\}$ and $\mathcal{V} = \mathcal{O} \cup \mathcal{U}'$, the family \mathcal{V} has the properties required by the theorem.

There are a number of corollaries available.

Corollary 1. *If a space X has a closure-preserving cover consisting of compact closed sets, then X is metacompact.*

There is an example in [2] of a completely regular, Hausdorff space, which has a closure-preserving cover consisting of finite sets, and hence is metacompact, yet which is not paracompact, nor even normal. However, by including an additional condition on the cover, we obtain a characterization of the class of locally compact paracompact spaces.

Corollary 2. *If a space X has a closure-preserving cover consisting of compact closed sets whose interiors cover X , then X is paracompact. In*

fact, the existence of such a cover is characteristic of those spaces which are both locally compact and paracompact.

It is interesting to note that these are paracompact spaces which do not admit even a σ -closure-preserving cover of compact sets. In [3] it is shown that the space of irrationals is such a space. The existence of a σ -closure-preserving cover implies another property, however.

Corollary 3. *If a space X has a cover which can be decomposed into countably many families, each of which is closure-preserving and consists of compact closed sets, then X is θ -refinable.*

Now we proceed to characterize certain types of spaces and covers. If X is a topological space and \mathcal{F} is a closure-preserving cover consisting of compact closed sets, then we introduce a partial ordering on X by agreeing that $x \leq y$ provided $K(x) \supset K(y)$.

Theorem 2. *If X is compact or σ -compact, then \mathcal{F} admits a closure-preserving cover of compact, closed sets with respect to which the ordering \leq is a linear ordering.*

Proof. If X is compact, let $\mathcal{F} = \{X\}$. Then for any points $x, y \in X$, we have $K(x) = K(y) = X$, and so it is certainly true that either $x \leq y$ or $y \leq x$. If X is a σ -compact space, let $\{G(i) | i \in \mathbb{Z}^+\}$ be a sequence of compact closed sets whose union is X . For each positive integer i , set $F(i) = \bigcup\{G(j) | j = 1, 2, \dots, i\}$. Then each set $F(i)$ is a compact, closed set, and $\mathcal{F} = \{F(i) | i \in \mathbb{Z}^+\}$ is a cover of X consisting of compact closed sets. Moreover, it is closure-preserving, for the union of any infinite subcollection of \mathcal{F} is the entire space X , and the union of any finite number of members of \mathcal{F} , say $F(i_1) \cup F(i_2) \cup \dots \cup F(i_n)$ is simply that set with the largest index from among i_1, i_2, \dots, i_n . In either case, we have a closed set. In addition, if \leq is defined as above, it is a linear order. Clearly, it is sufficient to prove that if $x, y \in X$, then $K(x) \subset K(y)$, or else $K(y) \subset K(x)$. Let j be the least of the integers $\{i | x \in F(i)\}$; let k be the least of the integers $\{i | y \in F(i)\}$, and assume that $j \leq k$. Then

$$K(x) = X - \bigcup\{F \in \mathcal{F} | x \notin F\} = X - \bigcup\{F(i) | i < j\}$$

and

$$K(y) = X - \bigcup\{F \in \mathcal{F} | y \notin F\} = X - \bigcup\{F(i) | i < k\}$$

so that $X - \bigcup\{F(i) | i < j\} \supset X - \bigcup\{F(i) | i < k\}$, that is, $K(x) \supset K(y)$, so that $x \leq y$, and \leq is seen to be a linear ordering.

The converse of the previous theorem is also true and we show this via a sequence of lemmas.

Lemma 5. *Let X be a topological space and let \mathcal{F} be a closure-preserving cover consisting of compact, closed sets. If \leq is a linear ordering and*

there exists a sequence in X which is cofinal with respect to \leq , then X is σ -compact.

Proof. Suppose that \leq is a linear ordering, that is, for $x, y \in X$, either $K(x) \supset K(y)$, or $K(x) \subset K(y)$. Let $Y = \{y(i) | i \in \mathbb{Z}^+\}$ be a cofinal sequence with respect to \leq . For each positive integer i , choose $F(i) \in \mathcal{F}$ such that $y(i) \in F(i)$. Since Y is cofinal with respect to \leq , for each $x \in X$, there is an integer i such that $x \leq y(i)$, that is, $K(x) \supset K(y(i))$. But then $y(i) \in F(i)$ and $K(x) \supset K(y(i))$ together imply that $x \in F(i)$, so we have that $X = \bigcup_{i \in \mathbb{Z}^+} F(i)$, and X is seen to be σ -compact.

Lemma 6. *Let X have the same properties as in Lemma 5. Then any infinite sequence in X is either cofinal with respect to \leq or has a limit point.*

Proof. Let $Y = \{y(i) | i \in \mathbb{Z}^+\}$ be an infinite sequence. If Y fails to be cofinal with respect to \leq , there is an $x \in X$ such that for each positive integer i , we have $y(i) \leq x$, that is $K(y(i)) \supset K(x)$. Choose $F \in \mathcal{F}$ such that $x \in F$. Then $K(y(i)) \supset K(x)$ and $x \in F$ together imply that $y(i) \in F$. But then Y is an infinite subset of the compact set F , hence must have a limit point.

Lemma 7. *If X is a topological space satisfying the hypotheses of Lemmas 5 and 6, then X is either σ -compact or countably compact.*

Proof. If X has even one cofinal sequence $Y = \{y(i) | i \in \mathbb{Z}^+\}$, then by Lemma 5, X is σ -compact. So we suppose that no sequence is cofinal. But then Lemma 6 guarantees that each sequence has a limit point, whence X is countably compact.

Theorem 3. *Let X be a topological space and let \mathcal{F} be a closure-preserving cover consisting of compact, closed sets. Suppose that \leq is a linear ordering, that is, for $x, y \in X$, either $K(x) \supset K(y)$, or $K(x) \subset K(y)$. Then X is either compact or σ -compact.*

Proof. By Lemma 7, X is either σ -compact or countably compact. By Corollary 1, X is metacompact. But it is well known that countable compactness and metacompactness together imply compactness, so X is either compact or σ -compact.

REFERENCES

1. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR 33 #1824.
2. H. Potoczny, *A nonparacompact space which admits a closure-preserving cover of compact sets*, Proc. Amer. Math. Soc. 32 (1972), 309–311. MR 44 #5923.
3. ———, *Closure-preserving families of compact sets*, General Topology and Appl. 3 (1973), 243–248. MR 48 #1166.
4. ———, *Closure-preserving families of finite sets*, Fund Math. (in press).
5. H. Tamano, *A characterization of paracompactness*, Fund. Math. 72 (1971), 189–201. MR 45 #5956.

6. R. Telgarsky, *Concerning two covering properties*, Proc. Japan Academy (in press).
7. ———, *Closure-preserving covers*, Fund. Math. **85** (1974), 165–175.
8. J. M. Worrell, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. **17** (1965), 820–830. MR 32 #427.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DAYTON, DAYTON, OHIO 45409

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, HELSINKI, FINLAND