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APPLICATIONS OF CLUSTER SETS IN MINIMAL TOPOLOGICAL SPACES

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ABSTRACT. Given a function f from a topological space X into a topological space Y and a point $x \in X$, *the cluster set of* f *at* x is $\mathcal{C}(f; x) = \bigcap\{\mathrm{Cl}(f(U)): U$ is a neighborhood of $x\}$, where $\mathrm{Cl}(U)$ denotes the closure of U. In this paper, Y is taken to be a minimal topological space and $\mathcal{C}(f; x)$ is used as a tool to obtain information about the continuity of f.

1. **Introduction.** If X is a topological space and $x \in X$, let $\mathfrak{N}(x)$ denote the nbd (neighborhood) system at x. Given a function f from a topological space X into a topological space Y, J. D. Weston [6] defined the *cluster set of* f *at* $x \in X$ to be

$$\mathcal{C}(f; x) = \bigcap\{\mathrm{Cl}(f(U)): U \in \mathfrak{N}(x)\}$$

where $\mathrm{Cl}(U)$ denotes the closure of U. Weston [6] observed that if Y is a Hausdorff space and f is continuous, then $\mathcal{C}(f; x)$ is degenerate. He also noted that the converse holds provided Y is compact, and, in general, does not hold if Y is not compact. In this paper we take Y to be either H-closed, minimal Hausdorff, minimal Urysohn, or minimal regular, and use $\mathcal{C}(f; x)$ as a tool to obtain information about the continuity of f.

2. **Preliminaries.** In this section we give some basic definitions and establish two lemmas that are useful in the sections which follow.

Let X be a topological space. An open subset U of X is said to be *regular open* [3, p. 92] if $U = \mathrm{Int}(\mathrm{Cl}(U))$, where Int denotes the interior operator. A subset F of X is said to be *regular closed* [3, p. 92] if $F = \mathrm{Cl}(\mathrm{Int}(F))$. If f is a function from X into a topological space Y, f is said to be *closed* (*almost closed*) if $f(K)$ is closed in Y for every closed (regular closed) set K in X. We say f is an *open* mapping if $f(V)$ is open in Y for every open set V in X.

Lemma 1. *Let f be a closed map from a regular space X into a space Y. If $f^{-1}(y)$ is closed in X for every $y \in Y$, then $\mathcal{C}(f; x)$ is degenerate for every $x \in X$.*

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Proof. The proof follows easily from the observation $\mathcal{C}(f;\, x) = \bigcap\{f(\mathrm{Cl}(U))\colon U \in \mathcal{N}(X)\}$.

Lemma 2. *Let f be an almost closed injection from a Hausdorff space X into a space Y, then $\mathcal{C}(f;\, x)$ is degenerate for every $x \in X$.*

Proof. The proof follows easily from the well-known fact [3, p. 92] that the closure of an open set is regular closed and the observation $\mathcal{C}(f;\, x) \subseteq \bigcap\{f(\mathrm{Cl}(U))\colon U \in \mathcal{N}(x)\}$.

3. ***H*-closed and minimal Hausdorff spaces.** Let f be a function from a space X into a space Y. We say f is *almost continuous* [4, Definition 3] at $x \in X$ if for every regular open nbd V of $f(x)$, there exists a $W \in \mathcal{N}(x)$ such that $f(W) \subseteq V$.

Theorem 3.1. *Let f be an open mapping from a space X into an H-closed space Y, and let $x \in X$. Then $\mathcal{C}(f;\, x)$ is degenerate if and only if f is almost continuous at x.*

Proof. *Necessity.* Let V be a regular open nbd of $f(x)$. Suppose $f(U) \cap (Y - V)$ is nonempty for every nbd U of x. Note that $(Y - V)$ is a regular closed subset of Y, and hence is H-closed [7, p. 127]. Now $\{\mathrm{Cl}(f(U)) \cap (Y - V)\colon U \in \mathcal{N}(x)\}$ is a family of closed sets, and their interiors with respect to $(Y - V)$ satisfy the finite intersection property. Thus $\mathcal{C}(f;\, x) \cap (Y - V)$ is nonempty and therefore $\mathcal{C}(f;\, x)$ is not degenerate.

Sufficiency. Assume f is almost continuous at x and let τ denote the topology on Y. Let τ_s denote the topology on Y generated by the regular open sets of τ. Now we have

$$\{f(x)\} = \bigcap\{\mathrm{Cl}_{\tau_s}(f(U))\colon U \in \mathcal{N}(x)\} \supseteq \bigcap\{\mathrm{Cl}_{\tau}(f(U))\colon U \in \mathcal{N}(x)\} \supseteq \mathcal{C}(f;\, x).$$

Hence $\mathcal{C}(f;\, x) = \{f(x)\}$ is degenerate.

Before stating the next theorem, we should point out that an example of an H-closed Urysohn space which is not compact may be found in [1, Example 3.13]. A function f from a space X into a space Y is said to be *connected* [5] if $f(C)$ is connected in Y for every connected subset C of X.

Theorem 3.2. *If f is an open connected mapping from a locally connected space X into an H-closed Urysohn space (Y, τ), then f is almost continuous at $x \in X$ if and only if $\mathcal{C}(f;\, x)$ is countable.*

Proof. *Sufficiency.* Assume $\mathcal{C}(f;\, x)$ is countable. Let $\mathcal{C}(x)$ be a nbd base of open connected sets at x. Note that $\mathrm{Cl}_\tau(f(U))$ is a regular closed connected set for every $U \in \mathcal{C}(x)$. Let τ_s denote the topology on Y generated by the regular open sets of τ. By Theorem 3.4(b) of [1], (Y, τ_s) is compact. Thus $\{\mathrm{Cl}_\tau(f(U))\colon U \in \mathcal{C}(x)\}$ is a collection of τ_s continua directed by inclusion, which implies $\mathcal{C}(f; x)$ is a τ_s continuum [7, Theorem 28.2, p. 203]. Consequently, $\mathcal{C}(f; x)$ must be either a single point or uncountable. Our assumption that $\mathcal{C}(f; x)$ is countable implies $\mathcal{C}(f; x)$ is a single point and, therefore, f is almost continuous by Theorem 3.1.

Necessity. Theorem 3.1.

We now focus our attention on minimal Hausdorff spaces, and at the end of this section state a theorem which gives results for H-closed and minimal Hausdorff spaces in combined form.

Theorem 3.3. *Let f be an open mapping from a space X into a minimal Hausdorff space Y. Then f is continuous at $x \in X$ if and only if $\mathcal{C}(f; X)$ is degenerate.*

Proof. We have only to show sufficiency. Assume $\mathcal{C}(f; x)$ is degenerate, and let $f(\mathfrak{N}) = \{f(U): U \in \mathfrak{N}(x)$ and U is open$\}$. Now $f(\mathfrak{N})$ is an open filterbase, and the assumption $\mathcal{C}(f; x)$ is degenerate implies $f(\mathfrak{N})$ has a unique adherent point which must be $f(x)$. Since Y is minimal Hausdorff, $f(\mathfrak{N})$ converges to $f(x)$ and therefore f is continuous.

We are now ready to apply Lemmas 1 and 2.

Theorem 3.4. *Let f be an open mapping from a space X into a minimal Hausdorff (H-closed) space Y.*

(1) *If X is regular, f is closed, and $f^{-1}(y)$ is closed for every $y \in Y$, then f is continuous (almost continuous).*

(2) *If f is an almost closed injection and X is Hausdorff, then f is continuous (almost continuous).*

Proof. (1) Lemma 1 and Theorem 3.3 (Theorem 3.1).

(2) Lemma 2 and Theorem 3.3 (Theorem 3.1).

Corollary 3.5. *Every continuous bijection from an H-closed space onto a Hausdorff space has an almost continuous inverse.*

Proof. The map f^{-1} is an open closed injection from a Hausdorff space into an H-closed space. Now apply Theorem 3.4(2).

4. **Minimal Urysohn and minimal regular spaces.** In order to be consistent with [1], our definition of regular (in this section only) includes the T_1 separation property. An open filterbase $\mathfrak{B}$ in a space is called a *Urysohn filterbase* if for each point $p \notin \bigcap\{\mathrm{Cl}(B): B \in \mathfrak{B}\}$, there is an open nbd U of p and $B \in \mathfrak{B}$ such that $\mathrm{Cl}(U) \cap \mathrm{Cl}(B) = \varnothing$. An open filterbase $\mathfrak{U}$ is called a *regular filterbase* if for each $U \in \mathfrak{U}$ there exists a $V \in \mathfrak{U}$ such that $\mathrm{Cl}(V) \subseteq U$. Note that a regular filterbase is a Urysohn filterbase. The following theorem gives a filterbase characterization of minimal regular and minimal Urysohn spaces.

Theorem 4.1 [1]. *A regular (Urysohn) space is minimal regular (Urysohn) if and only if every regular (Urysohn) filterbase with a unique adherent point converges.*

Theorem 4.2. *Let f be an open and closed mapping from a regular space X into a minimal regular (Urysohn) space Y. Then f is continuous at $x \in X$ if and only if $\mathcal{C}(f; x)$ is degenerate.*

Proof. We need only show sufficiency. Observe that for $x \in X$, $\{f(U): U \in \mathfrak{N}(x)\}$ is a regular filterbase (and hence a Urysohn filterbase) and apply Theorem 4.1.

We are now ready for another application of Lemma 1.

Theorem 4.3. *Let f be an open and closed mapping from a regular space X into a minimal regular (Urysohn) space Y. Then f is continuous if and only if the preimages of points in Y are closed in X.*

Proof. Necessity is obvious and sufficiency follows from Lemma 1 and Theorem 4.2.

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