A NOTE ON LIFTING BRAUER CHARACTERS

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ABSTRACT. A Brauer character of a finite group may be lifted to an ordinary character if it lies in a block whose defect groups are contained in a normal $p$-solvable subgroup.

By the Fong-Swan theorem [2, Theorem 72.1], an irreducible Brauer character of a finite $p$-solvable group $G$ may be lifted to an ordinary (complex) character of $G$. In other words, every Brauer character $\phi$ is the restriction of some ordinary character $\chi$ to the $p$-regular elements of $G$. Professor I. M. Isaacs has shown [5] that the character $\chi$ may be chosen to satisfy certain extra conditions which when $p$ is odd, uniquely determine $\chi$. By extending a theorem which appears in that paper [5, Theorem 3.1], the hypothesis of $p$-solvability on $G$ may be weakened somewhat.

Specifically, the main result of this paper is the following

Theorem. Let $\phi$ be an irreducible Brauer character of the finite group $G$, and assume that $\phi$ lies in a block whose defect groups are contained in a normal $p$-solvable subgroup of $G$. Then $\phi$ may be lifted to an ordinary character $\chi$ of $G$.

We will not be concerned with general uniqueness questions here.

For the remainder of this paper, $G$ denotes a finite group, and $F$ is a field of characteristic $p$ which is a splitting field for all subgroups of $G$. If $V$ is an $F[G]$-module, let $J(V)$ be the intersection of all maximal submodules of $V$. Finally, if $U$ and $V$ are $F[G]$-modules affording the Brauer characters $\phi$ and $\mu$, respectively, and if $V$ is irreducible, then the multiplicity of $\mu$ in $\phi$ is the multiplicity of $V$ as a composition factor of $U$.

Lemma 1. Let $N < G$ and let $W$ be an irreducible $F[N]$-module which affords the Brauer character $\mu$. Assume that $\mu$ can be lifted to an ordinary character $\Psi$ in such a way that the inertia groups $J_G(\mu)$ and $J_G(\Psi)$ coincide. Denote by $\mu^G$ the Brauer character which the induced module $W^G$ affords. Let $\mathcal{S}$ denote the set of all irreducible Brauer characters $\phi$ of $G$ which are constituents of $\mu^G$, but which are not afforded by any composition factor of $J(W^G)$. Finally, let $\mathcal{S}$ denote the set of ordinary irreducible characters $\chi$ of $G$ which are constituents of $\Psi^G$ and which have the property that the
decomposition number $d_\mathcal{S}$ does not vanish for some $\phi \in \mathcal{S}$. Then, restriction to $p$-regular elements is a 1-1 correspondence between the elements of $\mathcal{S}$ and the elements of $\mathcal{S}$.

**Proof.** Write $\mu^G = \sum_{\phi \in \mathcal{S}} f_\phi \phi + \Phi$, where no constituent of $\Phi$ lies in $\mathcal{S}$. We first compute the restriction of $\phi$ to $N$, where $\phi \in \mathcal{S}$. Let $W$ and $V$ be irreducible $F[N]$- and $F[G]$-modules affording $\mu$ and $\phi$, respectively. By Clifford's theorem, $V_N$ is completely reducible and, since $F$ is a splitting field for $N$, the multiplicity of $W$ as a composition factor of $V_N$ is the $F$-dimension of $\text{hom}_F[W, V_N]$. By the Nakayama relations [4, p. 556], this dimension equals the $F$-dimension of $\text{hom}_F[W^G, V]$. Since $V$ is irreducible, this last space is naturally isomorphic to $\text{hom}_F[W^G/W, V]$.

Since $W^G/J(W^G)$ is completely reducible, and $F$ is a splitting field for $G$, this last space has $F$-dimension equal to the multiplicity of $V$ in $W^G/J(W^G)$. However, $\phi \in \mathcal{S}$, which means that $V$ is not a composition factor of $J(W^G)$, so that the multiplicity of $V$ in $W^G/J(W^G)$ is $f_\phi$. Therefore, $\mu$ appears in $\phi_N$ with multiplicity $f_\phi$. We may write

$$\phi_N = f_\phi (\mu_1 + \cdots + \mu_t),$$

where $\mu = \mu_1$ and $\mu_1, \ldots, \mu_t$ are the distinct $G$-conjugates of $\mu$.

Similarly write $\Psi^G = \sum e_\chi \chi + X$, where each $\chi$ lies in $\mathcal{S}$, and no constituent of $X$ lies in $\mathcal{S}$. Since $\zeta^G_G(\mu) = \zeta^G_G(\Psi)$, the number of distinct $G$-conjugates of $\Psi$ is $t$, and by Frobenius reciprocity,

$$\chi_N = e_\chi (\Psi_1 + \cdots + \Psi_t),$$

where $\Psi = \Psi_1$, $\Psi_2, \ldots$, $\Psi_t$ are the distinct $G$-conjugates of $\Psi$.

Let $R$ and $S$ denote the set of $p$-regular elements of $G$ and $N$, respectively. We now use the equations

$$(\Psi|_S)^G = (\Psi^G|_S) \quad \text{and} \quad (\chi|_R)_N = (\chi_N|_S).$$

(The first equation follows from the fact that the values of $\mu^G$ may be computed by the usual formulas for an induced class function, a fact proved in [1, §25].) The first equation may be rewritten as

$$\sum_{\phi \in \mathcal{S}} f_\phi \phi + \Phi = \sum_{\chi \in \mathcal{S}} e_\chi \chi_R + \chi_R.$$

This implies that for $\chi \in \mathcal{S}$,

$$\chi_R = \sum_{\phi \in \mathcal{S}} d_\chi \phi + \eta_\chi,$$

where $\eta_\chi$ has constituents appearing in $\Phi$. Now, no $\phi$ in $\mathcal{S}$ appears as a constituent of $\chi_R$, so that

$$f_\chi = \sum_{\chi \in \mathcal{S}} e_\chi.$$

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For \( \chi \in \mathcal{I} \), the equation \( (\chi|_R)_N = (\chi_N)_S \) implies
\[
\sum_{\phi \in \mathcal{S}} d_{\chi, \phi} \phi_N + (\eta_\chi)_N = e_\chi (\mu_1 + \cdots + \mu_t),
\]
and since \( \phi_N = f_\phi (\mu_1 + \cdots + \mu_t) \), we get
\[
(**) \quad e_\chi \geq \sum_{\phi \in \mathcal{S}} d_{\chi, \phi} f_\phi.
\]

Now, combining (*) and (**):
\[
\int_\phi = \sum_{\chi \in \mathcal{I}} e_\chi d_{\chi, \phi} \geq \sum_{\chi \in \mathcal{I}} \sum_{\phi \in \mathcal{S}} d_{\chi, \phi'} f_{\phi'} d_{\chi, \phi} \\
= \sum_{\phi \in \mathcal{S}} \left( \sum_{\chi \in \mathcal{I}} d_{\chi, \phi'} d_{\chi, \phi} \right) f_{\phi'} \\
\geq \left( \sum_{\chi \in \mathcal{I}} d_{\chi, \phi}^2 \right) f_{\phi} \geq f_{\phi}.
\]

The last inequality is valid since (*) implies that, for every \( \phi \in \mathcal{S} \),
there exists \( \chi \in \mathcal{I} \) with \( d_{\chi, \phi} \neq 0 \). We now have that equality holds through-
out, in the above chain of inequalities, and, in particular, \( \sum_{\chi \in \mathcal{I}} d_{\chi, \phi}^2 = 1 \) holds
for every \( \phi \in \mathcal{S} \). Therefore, for every \( \phi \in \mathcal{S} \), there exists a unique \( \chi \in \mathcal{I} \)
with \( d_{\chi, \phi} = 1 \), and \( d_{\chi', \phi} = 0 \) for \( \chi' \neq \chi \) in \( \mathcal{I} \). But then (*) implies that \( f_{\phi} = e_\chi \), and since \( \mu(1) = \Psi(1) \), \( \chi \) must be a lift of \( \phi \). We have now proved that
every \( \phi \in \mathcal{S} \) has a unique lift \( \chi \) in \( \mathcal{I} \). Finally, if \( \chi \in \mathcal{I} \), then, by definition,
there exists \( \phi \in \mathcal{S} \) with \( d_{\chi, \phi} \neq 0 \). But, by the above, \( \chi \) is a lift of \( \phi \). Thus,
the map \( \chi \mapsto \chi_R \) is a 1-1 correspondence between \( \mathcal{I} \) and \( \mathcal{S} \).

**Definition.** Let \( V \) be an \( F[G] \)-module and \( N \) a subgroup of \( G \). \( V \) is \( N \)-
reducible if every exact sequence of \( F[G] \)-modules \( U \rightarrow V \rightarrow Y \), which splits
when considered as a sequence of \( F[N] \)-modules, necessarily splits as a se-
quence of \( F[G] \)-modules. Thus, every module is \( G \)-reducible, and a module
is 1-reducible iff it is completely reducible. (By using the other two posi-
tions of the exact sequence, one can define the usual notions of \( N \)-injectiv-
ity and \( N \)-projectivity.)

**Lemma 2.** Let \( V \) be an \( F[G] \)-module and \( N \) a subgroup of \( G \). Let \( T \) be
a set of coset representatives for the right cosets of \( N \) in \( G \). Assume that
there exists \( \alpha \in C_{F[G]}(N) \) such that \( \sum_{x \in T} x^{-1} \alpha x \) acts like the identity on
\( V \). Then \( V \) is \( N \)-reducible.

**Proof.** This is essentially the proof of (d) \( \rightarrow \) (a) of Theorem 1 of [3].

**Lemma 3.** Let \( N \triangleleft G \) and let \( \mu \) be an irreducible Brauer character of \( N \).
Assume that \( \mu \) can be lifted to an ordinary character \( \Psi \) with the property
that \( \Psi(\mu) = \Psi_G(\Psi) \). Finally, let \( B \) be a \( p \)-block of \( G \) whose defect groups
are contained in \( N \). Then the restriction to \( p \)-regular elements defines a
1-1 correspondence between the set of irreducible constituents of $\Psi^G$ which lie in $B$ and the set of irreducible Brauer constituents of $\mu^G$ which lie in $B$.

Proof. Define $\mathcal{S}$ and $\mathcal{T}$ as in the statement of Lemma 1, and again let $R$ denote the set of $p$-regular elements of $G$. Then $\chi \mapsto \chi_R$ is a 1-1 correspondence between the elements of $\mathcal{S}$ and $\mathcal{T}$. Clearly $\chi \in B$ iff $\chi_R \in B$. It suffices to show that all the irreducible Brauer constituents of $\mu^G$ which belong to $B$ necessarily lie in $\mathcal{S}$.

Let $W$ afford $\mu$ and let $e$ denote the centrally primitive idempotent of $F[G]$ which corresponds to the block $B$. Then $W^G = (W^G)_e \cdot (W^G)(1 - e)$. The composition factors of $(W^G)_e$ afford Brauer characters in $B$, and no composition factor of $(W^G)(1 - e)$ belongs to $B$. We must show that $J(W^G)_e \subseteq (W^G)(1 - e)$, and this is equivalent to the statement that $(W^G)_e$ is completely reducible.

Since $B$ has a defect group contained in $N$, it follows that the block idempotent $e$ has a representation of the form $e = \sum_{x \in T} x^{-1} \alpha x$, where $T$ is a set of coset representatives for $N$ in $G$, and $\alpha \in C_{F[G]}(N)$. (This is essentially Lemma 54.8 of [2, p. 346] with $F$ in place of $R$.)

Thus, $(W^G)_e$ is $N$-reducible by Lemma 2. Therefore, $(W^G)_e$ is completely reducible as an $F[G]$-module iff $(W^G)_e|_N$ is completely reducible as an $F[N]$-module. However $(W^G)_e|_N$ is a summand of $(W^G)_N = \sum_{x \in T} W \otimes x$ where each $W \otimes x$ is simple. Hence, $(W^G)_e$ is completely reducible, and we are done.

In order to prove the main theorem of this paper, we need the strengthened version of the Fong-Swan theorem appearing in [5].

Lemma 1. Let $N$ be a $p$-solvable group and $\mu$ an irreducible Brauer character of $N$. Then there exists an ordinary irreducible character $\Psi$ of $N$ which lifts $\mu$ and satisfies $\delta_A(\mu) = \delta_A(\Psi)$, where $A$ is the automorphism group of $N$.

Proof. This is contained in Theorem 5.4 of [5].

We now present a proof of the theorem quoted at the beginning of the paper.

Let $\phi$ be an irreducible Brauer character of $G$ and assume that $\phi$ lies in a block whose defect groups are contained in the normal $p$-solvable subgroup $N$. Let $\mu$ be a constituent of $\phi|_N$ and lift $\mu$ to an ordinary character $\Psi$ satisfying the conclusion of Lemma 4. Since $G$ induces on $N$ a group of automorphisms, clearly $\delta_G(\mu) = \delta_G(\Psi)$. Lemma 3 now implies that $\phi$ has a lift (which is a constituent of $\Psi^G$).

We remark that by replacing $N$ by the largest normal $p$-solvable subgroup of $G$ (so as to assume that $N$ is characteristic in $G$), it is easy to show that $\phi$ has a lift $\chi$ which satisfies $\delta_G(\phi) = \delta_{\text{aut}(G)}(\chi)$. 

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Closing remarks. I would like to thank the referee for his careful reading of this paper and for pointing out that, in the situation of Lemma 1, the set $\mathcal{S}$ consists precisely of those irreducible Brauer constituents of $\mu^G$ which have vertices contained in $N$. (Of course, the vertex of a Brauer character means the vertex of an irreducible $F[G]$-module which affords it.) This observation follows from the equivalence of (i) and (iii) in the following

**Proposition.** Let $N \triangleleft G$, $W$ an irreducible $F[N]$-module and $V$ an irreducible $F[G]$-module. Assume that $V$ is a composition factor of $W^G$. Then the following conditions on $V$ are equivalent.

(i) A vertex for $V$ is contained in $N$.

(ii) $W^G = U \downarrow S$ where no composition factor of $U$ is isomorphic to $V$, and $S$ is a direct sum of simple modules all being isomorphic to $V$.

(iii) $V$ is not a composition factor of $J(W^G)$.

**Proof.** (i) $\implies$ (ii). Write $W^G = U \downarrow S$ with $S$ isomorphic to a direct sum of copies of $V$ and with $\dim F U$ minimal. Suppose $X$ and $Y$ are submodules of $U$ with $Y \subseteq X$ and $X/Y \cong V$. Then $Y \hookrightarrow X \twoheadrightarrow V$ is an exact sequence of $F[G]$-modules. Since $X$ is a submodule of $W^G$, and $W^G|_N$ is completely reducible, the sequence splits when regarded as a sequence of $F[N]$-modules. However, a vertex of $V$ is contained in $N$, so $V$ is $N$-projective and the sequence splits as a sequence of $F[G]$-modules. Thus $V$ is isomorphic to a submodule $V'$ of $X$ and, hence, of $U$. By considering the sequence $V' \hookrightarrow U \twoheadrightarrow U/V'$ and using the fact that $V'$ is $N$-injective, we have $U = U_0 \downarrow V_1$. But then $W^G = U_0 \downarrow (V_1 \downarrow S)$, contradicting the minimality of $\dim F U$, and thereby proving (ii).

(ii) $\implies$ (iii). The hypothesis of (ii) implies $J(W^G) \subseteq U$ and (iii) is immediate.

(iii) $\implies$ (i). Since $V$ is a composition factor of $W^G$, it follows from the Nakayama relations and the semisimplicity of $V_N$ that $W$ is a summand (and, hence, a homomorphic image) of $V_N$. By the Nakayama relations again, $V$ is isomorphic to a submodule, say $V_1$, of $W^G$. Since $V_1 \not\subseteq J(W^G)$, we have $W^G = V_1 \downarrow M$ for some maximal submodule $M$ of $W^G$. This equation implies that $V_1$ is $N$-projective, and so $N$ contains a vertex of $V_1$, proving (i).

As the referee has kindly pointed out, this Proposition, together with Lemmas 1 and 4 imply the following strengthened version of the main theorem of this paper:

**Theorem.** Let $\phi$ be an irreducible Brauer character of $G$ and assume that a vertex for a module affording $\phi$ is contained in a normal $p$-solvable subgroup of $G$. Then $\phi$ may be lifted to an ordinary character $\chi$ of $G$.

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