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A NOTE ON LIFTING BRAUER CHARACTERS

STEPHEN M. GAGOLA, JR.

ABSTRACT. A Brauer character of a finite group may be lifted to an ordinary character if it lies in a block whose defect groups are contained in a normal p-solvable subgroup.

By the Fong-Swan theorem [2, Theorem 72.1], an irreducible Brauer character of a finite p-solvable group G may be lifted to an ordinary (complex) character of G. In other words, every Brauer character ϕ is the restriction of some ordinary character χ to the p-regular elements of G. Professor I. M. Isaacs has shown [5] that the character χ may be chosen to satisfy certain extra conditions which when p is odd, uniquely determine χ. By extending a theorem which appears in that paper [5, Theorem 3.1], the hypothesis of p-solvability on G may be weakened somewhat.

Specifically, the main result of this paper is the following

Theorem. *Let ϕ be an irreducible Brauer character of the finite group G, and assume that ϕ lies in a block whose defect groups are contained in a normal p-solvable subgroup of G. Then ϕ may be lifted to an ordinary character χ of G.*

We will not be concerned with general uniqueness questions here.

For the remainder of this paper, G denotes a finite group, and F is a field of characteristic p which is a splitting field for all subgroups of G. If V is an $F[G]$-module, let $J(V)$ be the intersection of all maximal submodules of V. Finally, if U and V are $F[G]$-modules affording the Brauer characters ϕ and μ, respectively, and if V is irreducible, then the multiplicity of μ in ϕ is the multiplicity of V as a composition factor of U.

Lemma 1. *Let $N \lhd G$ and let W be an irreducible $F[N]$-module which affords the Brauer character μ. Assume that μ can be lifted to an ordinary character Ψ in such a way that the inertia groups $\mathcal{I}_G(\mu)$ and $\mathcal{I}_G(\Psi)$ coincide. Denote by μ^G the Brauer character which the induced module W^G affords. Let $\mathcal{S}$ denote the set of all irreducible Brauer characters ϕ of G which are constituents of μ^G, but which are not afforded by any composition factor of $J(W^G)$. Finally, let $\mathcal{T}$ denote the set of ordinary irreducible characters χ of G which are constituents of Ψ^G and which have the property that the*

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decomposition number $d_{\chi\phi}$ does not vanish for some $\phi \in \mathcal{S}$. Then, restriction to p-regular elements is a 1-1 *correspondence between the elements of $\mathcal{T}$ and the elements of $\mathcal{S}$.*

Proof. Write $\mu^G = \Sigma_{\phi \in \mathcal{S}} f_\phi \phi + \Phi$, where no constituent of Φ lies in $\mathcal{S}$. We first compute the restriction of ϕ to N, where $\phi \in \mathcal{S}$. Let W and V be irreducible $F[N]$- and $F[G]$-modules affording μ and ϕ, respectively. By Clifford's theorem, V_N is completely reducible and, since F is a splitting field for N, the multiplicity of W as a composition factor of V_N is the F-dimension of $\hom_{F[N]}(W, V_N)$. By the Nakayama relations [4, p. 556], this dimension equals the F-dimension of $\hom_{F[G]}(W^G, V)$. Since V is irreducible, this last space is naturally isomorphic to $\hom_{F[G]}(W^G/J(W^G), V)$.

Since $W^G/J(W^G)$ is completely reducible, and F is a splitting field for G, this last space has F-dimension equal to the multiplicity of V in $W^G/J(W^G)$. However, $\phi \in \mathcal{S}$, which means that V is not a composition factor of $J(W^G)$, so that the multiplicity of V in $W/J(W^G)$ is f_ϕ. Therefore, μ appears in ϕ_N with multiplicity f_ϕ. We may write

$$\phi_N = f_\phi(\mu_1 + \cdots + \mu_t),$$

where $\mu = \mu_1$ and $\mu_1, \ldots, \mu_t$ are the distinct G-conjugates of μ.

Similarly write $\Psi^G = \Sigma\, e_\chi \chi + X$, where each χ lies in $\mathcal{T}$, and no constituent of X lies in $\mathcal{T}$. Since $\mathcal{I}_G(\mu) = \mathcal{I}_G(\Psi)$, the number of distinct G-conjugates of Ψ is t, and by Frobenius reciprocity,

$$\chi_N = e_\chi(\Psi_1 + \cdots + \Psi_t),$$

where $\Psi = \Psi_1, \Psi_2, \ldots, \Psi_t$ are the distinct G-conjugates of Ψ.

Let R and S denote the set of p-regular elements of G and N, respectively. We now use the equations

$$(\Psi|_S)^G = (\Psi^G)_R \quad \text{and} \quad (\chi|_R)_N = (\chi_N)|_S.$$

(The first equation follows from the fact that the values of μ^G may be computed by the usual formulas for an induced class function, a fact proved in [1, §25].) The first equation may be rewritten as

$$\sum_{\phi\in\mathcal{S}} f_\phi \phi + \Phi = \sum_{\chi\in\mathcal{T}} e_\chi \chi_R + X_R.$$

This implies that for $\chi \in \mathcal{T}$,

$$\chi_R = \sum_{\phi\in\mathcal{S}} d_{\chi\phi}\phi + \eta_\chi,$$

where η_χ has constituents appearing in Φ. Now, no ϕ in $\mathcal{S}$ appears as a constituent of X_R, so that

$$(*) \qquad f_\phi = \sum_{\chi\in\mathcal{T}} e_\chi d_{\chi\phi}.$$

For $\chi \in \mathcal{T}$, the equation $(\chi|_R)_N = (\chi_N)|_S$ implies

$$\sum_{\phi \in \mathcal{S}} d_{\chi\phi}\phi_N + (\eta_\chi)_N = e_\chi(\mu_1 + \cdots + \mu_t),$$

and since $\phi_N = f_\phi(\mu_1 + \cdots + \mu_t)$, we get

$$(**) \qquad e_\chi \geq \sum_{\phi \in \mathcal{S}} d_{\chi\phi} f_\phi.$$

Now, combining $(*)$ and $(**)$:

$$\begin{aligned} f_\phi &= \sum_{\chi \in \mathcal{T}} e_\chi d_{\chi\phi} \geq \sum_{\chi \in \mathcal{S}} \sum_{\phi' \in \mathcal{S}} d_{\chi\phi'} f_{\phi'} d_{\chi\phi} \\ &= \sum_{\phi' \in \mathcal{S}} \Big(\sum_{\chi \in \mathcal{T}} d_{\chi\phi'} d_{\chi\phi} \Big) f_{\phi'} \\ &\geq \Big(\sum_{\chi \in \mathcal{T}} d_{\chi\phi}^2 \Big) f_\phi \geq f_\phi. \end{aligned}$$

The last inequality is valid since $(*)$ implies that, for every $\phi \in \mathcal{S}$, there exists $\chi \in \mathcal{T}$ with $d_{\chi\phi} \neq 0$. We now have that equality holds throughout, in the above chain of inequalities, and, in particular, $\Sigma_{\chi \in \mathcal{T}} d_{\chi\phi}^2 = 1$ holds for every $\phi \in \mathcal{S}$. Therefore, for every $\phi \in \mathcal{S}$, there exists a unique $\chi \in \mathcal{T}$ with $d_{\chi\phi} = 1$, and $d_{\chi'\phi} = 0$ for $\chi' \neq \chi$ in $\mathcal{T}$. But then $(*)$ implies that $f_\phi = e_\chi$, and since $\mu(1) = \Psi(1)$, χ must be a lift of ϕ. We have now proved that every $\phi \in \mathcal{S}$ has a unique lift χ in $\mathcal{T}$. Finally, if $\chi \in \mathcal{T}$, then, by definition, there exists $\phi \in \mathcal{S}$ with $d_{\chi\phi} \neq 0$. But, by the above, χ is a lift of ϕ. Thus, the map $\chi \mapsto \chi_R$ is a 1-1 correspondence between $\mathcal{T}$ and $\mathcal{S}$.

Definition. Let V be an $F[G]$-module and N a subgroup of G. V is N-*reducible* if every exact sequence of $F[G]$-modules $U \rightarrowtail V \twoheadrightarrow Y$, which splits when considered as a sequence of $F[N]$-modules, necessarily splits as a sequence of $F[G]$-modules. Thus, every module is G-reducible, and a module is 1-reducible iff it is completely reducible. (By using the other two positions of the exact sequence, one can define the usual notions of N-injectivity and N-projectivity.)

Lemma 2. *Let V be an $F[G]$-module and N a subgroup of G. Let T be a set of coset representatives for the right cosets of N in G. Assume that there exists $\alpha \in C_{F[G]}(N)$ such that $\Sigma_{x \in T}\, x^{-1}\alpha x$ acts like the identity on V. Then V is N-reducible.*

Proof. This is essentially the proof of $(d) \longrightarrow (a)$ of Theorem 1 of [3].

Lemma 3. *Let $N \lhd G$ and let μ be an irreducible Brauer character of N. Assume that μ can be lifted to an ordinary character Ψ with the property that $\mathcal{I}_G(\mu) = \mathcal{I}_G(\Psi)$. Finally, let B be a p-block of G whose defect groups are contained in N. Then the restriction to p-regular elements defines a*

1-1 *correspondence between the set of irreducible constituents of* Ψ^G *which lie in* B *and the set of irreducible Brauer constituents of* μ^G *which lie in* B.

Proof. Define $\mathcal{S}$ and $\mathcal{T}$ as in the statement of Lemma 1, and again let R denote the set of p-regular elements of G. Then $\chi \mapsto \chi_R$ is a 1-1 correspondence between the elements of $\mathcal{S}$ and $\mathcal{T}$. Clearly $\chi \in B$ iff $\chi_R \in B$. It suffices to show that all the irreducible Brauer constituents of μ^G which belong to B necessarily lie in $\mathcal{S}$.

Let W afford μ and let e denote the centrally primitive idempotent of $F[G]$ which corresponds to the block B. Then $W^G = (W^G)e \dotplus (W^G)(1-e)$. The composition factors of $(W^G)e$ afford Brauer characters in B, and no composition factor of $(W^G)(1-e)$ belongs to B. We must show that $J(W^G) \subseteq (W^G)(1-e)$, and this is equivalent to the statement that $(W^G)e$ is completely reducible.

Since B has a defect group contained in N, it follows that the block idempotent e has a representation of the form $e = \sum_{x \in T} x^{-1}\alpha x$, where T is a set of coset representatives for N in G, and $\alpha \in C_{F[G]}(N)$. (This is essentially Lemma 54.8 of [2, p. 346] with F in place of R.)

Thus, $(W^G)e$ is N-reducible by Lemma 2. Therefore, $(W^G)e$ is completely reducible as an $F[G]$-module iff $(W^G)e|_N$ is completely reducible as an $F[N]$-module. However $(W^G)e|_N$ is a summand of $(W^G)_N = \sum_{x \in T} \cdot W \otimes x$ where each $W \otimes x$ is simple. Hence, $(W^G)e$ is completely reducible, and we are done.

In order to prove the main theorem of this paper, we need the strengthened version of the Fong-Swan theorem appearing in [5].

Lemma 4. *Let* N *be a* p*-solvable group and* μ *an irreducible Brauer character of* N. *Then there exists an ordinary irreducible character* Ψ *of* N *which lifts* μ *and satisfies* $\mathcal{I}_A(\mu) = \mathcal{I}_A(\Psi)$, *where* A *is the automorphism group of* N.

Proof. This is contained in Theorem 5.4 of [5].

We now present a proof of the theorem quoted at the beginning of the paper.

Let ϕ be an irreducible Brauer character of G and assume that ϕ lies in a block whose defect groups are contained in the normal p-solvable subgroup N. Let μ be a constituent of ϕ_N and lift μ to an ordinary character Ψ satisfying the conclusion of Lemma 4. Since G induces on N a group of automorphisms, clearly $\mathcal{I}_G(\mu) = \mathcal{I}_G(\Psi)$. Lemma 3 now implies that ϕ has a lift (which is a constituent of Ψ^G).

We remark that by replacing N by the largest normal p-solvable subgroup of G (so as to assume that N is characteristic in G), it is easy to show that ϕ has a lift χ which satisfies $\mathcal{I}_{\mathrm{Aut}(G)}(\phi) = \mathcal{I}_{\mathrm{Aut}(G)}(\chi)$.

Closing remarks. I would like to thank the referee for his careful reading of this paper and for pointing out that, in the situation of Lemma 1, the set $\mathcal{S}$ consists precisely of those irreducible Brauer constituents of μ^G which have vertices contained in N. (Of course, the vertex of a Brauer character means the vertex of an irreducible $F[G]$-module which affords it.) This observation follows from the equivalence of (i) and (iii) in the following

Proposition. *Let $N \lhd G$, W an irreducible $F[N]$-module and V an irreducible $F[G]$-module. Assume that V is a composition factor of W^G. Then the following conditions on V are equivalent.*

(i) *A vertex for V is contained in N.*

(ii) *$W^G = U \dotplus S$ where no composition factor of U is isomorphic to V, and S is a direct sum of simple modules all being isomorphic to V.*

(iii) *V is not a composition factor of $J(W^G)$.*

Proof. (i) $\to$ (ii). Write $W^G = U \dotplus S$ with S isomorphic to a direct sum of copies of V and with $\dim_F U$ minimal. Suppose X and Y are submodules of U with $Y \leq X$ and $X/Y \cong V$. Then $Y \rightarrowtail X \twoheadrightarrow V$ is an exact sequence of $F[G]$-modules. Since X is a submodule of W^G, and $W^G|_N$ is completely reducible, the sequence splits when regarded as a sequence of $F[N]$-modules. However, a vertex of V is contained in N, so V is N-projective and the sequence splits as a sequence of $F[G]$-modules. Thus V is isomorphic to a submodule V_1 of X and, hence, of U. By considering the sequence $V_1 \rightarrowtail U \twoheadrightarrow U/V_1$ and using the fact that V_1 is N-injective, we have $U = U_0 \dotplus V_1$. But then $W^G = U_0 \dotplus (V_1 \dotplus S)$, contradicting the minimality of $\dim_F U$, and thereby proving (ii).

(ii) $\to$ (iii). The hypothesis of (ii) implies $J(W^G) \subseteq U$ and (iii) is immediate.

(iii) $\to$ (i). Since V is a composition factor of W^G, it follows from the Nakayama relations and the semisimplicity of V_N that W is a summand (and, hence, a homomorphic image) of V_N. By the Nakayama relations again, V is isomorphic to a submodule, say V_1, of W^G. Since $V_1 \not\subseteq J(W^G)$, we have $W^G = V_1 \dotplus M$ for some maximal submodule M of W^G. This equation implies that V_1 is N-projective, and so N contains a vertex of V_1, proving (i).

As the referee has kindly pointed out, this Proposition, together with Lemmas 1 and 4 imply the following strengthened version of the main theorem of this paper:

Theorem. *Let ϕ be an irreducible Brauer character of G and assume that a vertex for a module affording ϕ is contained in a normal p-solvable subgroup of G. Then ϕ may be lifted to an ordinary character χ of G.*

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824