PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

NORMAL SUBGROUPS CONTAINED IN THE FRATTINI SUBGROUP. II

W. MACK HILL

ABSTRACT. If p is an odd prime and H is a p-group with a characteristic subgroup K such that $|K| > |K \cap Z(H)| = p$, then H cannot be a normal subgroup contained in the Frattini subgroup of any finite group G.

We consider only finite groups. The order of the group G is $|G|$, $Z(G)$ is the center of G, $A(G)$ is the automorphism group of G and $I(G)$ is the group of inner automorphisms. If G is nilpotent, $\mathrm{cl}(G)$ denotes its nilpotence class. Other notation is also standard.

Our aim is to prove the following

Theorem. *Let H be a p-group, p an odd prime, with a characteristic subgroup K such that $|K| > |K \cap Z(H)| = p$. Then H cannot be a normal subgroup contained in the Frattini subgroup of any finite group G.*

This result appears in [6] for arbitrary prime p, but under the additional hypothesis that $\mathrm{cl}(K) \neq 2$. It appears in [3] for the case that p is any prime and G is p-supersolvable. The case that $|H| = |K| = p^3$ is covered in [5].

With no loss of generality (see [6]), we take $K = H$ and $\mathrm{cl}(H) = 2$. Then H is extra-special. For a discussion of extra-special p-groups and their automorphisms the reader is referred to [1], [7] and [8].

Our argument is based on two lemmas, the first of which is mentioned in [2]. (The author is grateful to Professor David Goldschmidt for a very helpful conversation concerning this result.)

Lemma 1. *If H is an extra-special p-group of exponent p, p odd, then $A(H)$ splits over $I(H)$.*

Proof. $H = \langle x_1, x_2, \cdots, x_n, z\rangle$ with $x_i^p = z^p = 1$ for each i and $[x_1, x_2] = [x_3, x_4] = \cdots = [x_{n-1}, x_n] = z$. Further, $[x_i, x_j] = 1$ unless $\{i, j\}$ is one of $\{1, 2\}, \{3, 4\}, \cdots, \{n-1, n\}$. Each element of H has unique representation as $(\Pi_{i=1}^n x_i^{a_i})z^b$ with $0 \le a_i, b < p$.

If $\sigma \in A(H)$, then for each i, $\sigma(x_i) = (\Pi_{j=1}^n x_j^{a_{ij}})z^{b_i}$ with $(a_{ij}) \in \mathrm{GL}(n, p)$ and $0 \le b_i < p$. Further, $\sigma \in I(H)$ if and only if (a_{ij}) is the identity matrix.

Received by the editors December 16, 1974.

AMS (MOS) subject classifications (1970). Primary 20D25; Secondary 20D15, 20D45.

Key words and phrases. Extra-special p-group, Frattini subgroup.

Now the mapping τ of $\{x_1, x_2, \cdots, x_n, z\}$ into H, defined by $\tau(x_i) = x_i^{-1}$ $(i = 1, 2, \cdots, n)$ and $\tau(z) = z$, determines an automorphism $\tau \in A(H)$, and $C_{A(H)}(\tau)$ has trivial intersection with $I(H)$. Let σ map H into H and γ map $\{x_1, x_2, \cdots, x_n\}$ into H and suppose that for $i = 1, 2, \cdots, n$,

$$\sigma(x_i) = \left(\prod_{j=1}^{n} x_j^{a_{ij}}\right) z^{b_i} \quad \text{and} \quad \gamma(x_i) = \left(\prod_{j=1}^{n} x_j^{a_{ij}}\right) z^{c_i}.$$

Consider the system of linear congruences

$$\sum_{j=1}^{n} a_{ij} t_j \equiv c_i - b_i \pmod p, \qquad i = 1, 2, \ldots, n.$$

If (a_{ij}) is nonsingular, there exists a unique solution $(d_1, d_2, \cdots, d_n)$ with $0 \le d_i < p$. The mapping ρ of $\{x_1, x_2, \cdots, x_n, z\}$ into H defined by $\rho(x_i) = x_i z^{d_i}$ $(i = 1, 2, \cdots, n)$ and $\rho(z) = z$ determines an inner automorphism $\rho \in I(H)$ and

$$\rho\sigma(x_i) = \rho\left[\left(\prod_{j=1}^{n} x_j^{a_{ij}}\right) z^{b_i}\right] = \left(\prod_{j=1}^{n} x_j^{a_{ij}}\right) z^{e_i} = \left(\prod_{j=1}^{n} x_j^{a_{ij}}\right) z^{c_i} = \gamma(x_i)$$

where $e_i = \sum_{j=1}^{n} a_{ij} d_j + b_i$. In particular, if $\sigma \in A(H)$, then γ agrees with $\rho\sigma$ on the generating set $\{x_1, x_2, \cdots, x_n\}$ and, hence, determines an automorphism $\gamma \in A(H)$ with $\rho\sigma = \gamma$.

We now show that for arbitrary $\sigma \in A(H)$, the exponents c_i $(i = 1, 2, \cdots, n)$ above can be selected so that $\gamma \in C_{A(H)}(\tau)$. For $i = 1, 2, \cdots, n$ let c_i be the unique solution of the linear congruence

$$2t + f_i = 2t + \sum_{k=1}^{n-1} a_{ik} a_{i(k+1)} \equiv 0 \pmod p.$$

Then

$$\gamma\tau(x_i) = \gamma(x_i^{-1}) = \left(\prod_{j=1}^{n} x_j^{a_{ij}}\right)^{-1} z^{-c_i} = \left(\prod_{j=1}^{n} x_j^{-a_{ij}}\right) z^{-c_i - f_i}$$

$$= \left(\prod_{j=1}^{n} x_j^{-a_{ij}}\right) z^{c_i} = \tau\left[\left(\prod_{j=1}^{n} x_j^{a_{ij}}\right) z^{c_i}\right] = \tau\gamma(x_i).$$

Thus, for each $\sigma \in A(H)$, there exists $\rho \in I(H)$ and $\gamma \in C_{A(H)}(\tau)$ such that $\sigma = \rho^{-1}\gamma$, i.e. $A(H) = I(H) C_{A(H)}(\tau)$. Hence, $C_{A(H)}(\tau)$ complements $I(H)$ in $A(H)$, completing the proof of Lemma 1.

Lemma 2. *If H is an extra-special p-group of exponent p^2, p odd, then H has a characteristic subgroup K of order p^2.*

Proof. $H = \langle x_1, x_2, \cdots, x_n, z\rangle$ with $x_1^{p^2} = x_i^p = z^p = 1$ $(i = 2, 3, \cdots, n)$, $x_1^p = z$ and $[x_1, x_2] = [x_3, x_4] = \cdots = [x_{n-1}, x_n] = z$. Further, $[x_i, x_j] = 1$ unless $\{i, j\}$ is one of $\{1, 2\}, \{3, 4\}, \cdots, \{n-1, n\}$. The subgroup $\langle z, x_2, \cdots, x_n\rangle$, consisting precisely of those elements in H satisfying $h^p = 1$, is characteristic in H, and its center, also characteristic in H, is

$\langle z, x_2 \rangle$, of order p^2. Take $K = \langle z, x_2 \rangle$. This proves Lemma 2.

Now, let H be an extra-special p-group, p odd. If the exponent of H is p, then Lemma 1 together with III. 3.2 and III. 3.13 of [7] implies that H cannot be a normal subgroup contained in the Frattini subgroup of any finite group G. If on the other hand the exponent of H is p^2, then H has a characteristic subgroup K of order p^2 (Lemma 2), which of necessity intersects $Z(H)$ in a subgroup of order p. By [6], the desired conclusion follows, and the proof of the theorem is complete.

For the case $p = 2$, we have very little information. Again we lose no generality by taking $H = K$ and $\mathrm{cl}(H) = 2$. Thus, as before, H is extra-special. From (2), the splitting of $A(H)$ over $I(H)$ occurs for extra-special 2-groups of orders 2^3 and 2^5 and does not occur for those of order 2^7 and larger. Hence, a 2-group H with characteristic subgroup K of order 2^3 or 2^5 and intersecting $Z(H)$ in a subgroup of order 2 cannot be a normal subgroup contained in the Frattini subgroup of any finite group G. Since the splitting of $A(H)$ over $I(H)$ is only a sufficient condition for the above nonembeddability conclusion, the question remains open for extra-special 2-groups of larger orders.

Added in proof. Professor Homer Bechtell has observed that Griess' work (2) can be used to show that if H is an extra-special 2-group of order larger than 32, there exists a (nonsolvable) group G having Frattini subgroup H.

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DEPARTMENT OF MATHEMATICS, WORCESTER STATE COLLEGE, WORCESTER, MASSACHUSETTS 01602