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ON LAKSHMIKANTHAM'S COMPARISON FOR ORDINARY DIFFERENTIAL EQUATIONS

GIOVANNI VIDOSSICH

ABSTRACT. The paper describes the relation between Lakshmikantham's comparison and other known facts.

V. Lakshmikantham introduced in [4] a new setting for the comparison of ordinary differential equations. The aim of this paper is to show that Lakshmikantham's comparison is a special case, in the sense explained by the statement of the theorem below, of the pairing comparison considered in Vidossich [6]. The implications of this discovery are: (i) all the results in the framework of Lakshmikantham's pairing can be generalized by setting them in the framework of the pairing comparison; (ii) the stability theorem Lakshmikantham and Leela [5, Theorem 2.13.1] claimed as new is in reality a special case of an older one [5, Theorem 2.13.3] which appeared in Conti and Sansone [2].

The above claims are consequences of the theorem below. Concerning (i), we avoid carrying out the program since the interested reader can do it easily; the proofs of the generalizations can be based on the existing arguments of the theorems to be generalized simply by substituting $\|\cdot\|$ by $\|\cdot\|^2$. We refer to Lakshmikantham and Leela [5], Ladas and Lakshmikantham [3] and Becker and Vidossich [1, Theorem 4] for the related results and bibliography.

Theorem. *Let X be a Banach space, $A \subseteq \mathbf{R} \times X$, $B \subseteq \mathbf{R}^2$ and $f, g: A \to X$, $\omega: B \to \mathbf{R}$ continuous functions. Let $O:]0, \epsilon] \to \mathbf{R}$ be such that*

$$\lim_{h \downarrow 0} \frac{O(h)}{h} = 0.$$

Then each one of the following statements,

(1) $$\|x + hf(t, x)\| \le \|x\| + h\omega(t, \|x\|) + O(h),$$

(2) $$\|x - y + h(f(t, x) - f(t, y))\| \le \|x - y\| + h\omega(t, \|x - y\|) + O(h),$$

(3) $$\|x - y + h(f(t, x) - g(t, y))\| \le \|x - y\| + h\omega(t, \|x - y\|) + O(h),$$

implies the corresponding one of the following statements,

(1)* $$(f(t, x), x)_- \le \omega(t, \|x\|)\|x\|,$$

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$$(2)^* \qquad (f(t, x) - f(t, y), x - y)_- \leq \omega(t, \|x - y\|)\|x - y\|,$$

$$(3)^* \qquad (f(t, x) - g(t, y), x - y)_- \leq \omega(t, \|x - y\|)\|x - y\|,$$

but the converse fails. In other words, we have

$$(i) \Rightarrow (i)^* \not\Rightarrow (i) \qquad (i = 1, 2, 3).$$

In the above statement, $(\cdot,\cdot)_-$ denotes the generalized inner product on a Banach space X,

$$(x, y)_- = \inf\{h(x) | h \in J(y)\},$$

where $J : X \longrightarrow 2^{X^*}$ is the duality map

$$J(x) = \{h \in X^* | \|h\| = \|x\|, h(x) = \|x\|^2\}.$$

When X is a Hilbert space, $(\cdot,\cdot)_-$ coincides with the inner product.

Proof of Theorem. First we note the following property of the generalized pairing

$$(*) \qquad (x + y, x)_- = \|x\|^2 + (y, x)_- .$$

For, choosing $h \in J(x)$, we have

$$h(x + y) = h(x) + h(y) = \|x\|^2 + h(y)$$

from which we have $(*)$ by taking $\inf_h$.

$(1) \Rightarrow (1)^*$. From (1) it follows that

$$h\omega(t, \|x\|) + O(h) \geq \|x + hf(t, x)\| - \|x\|.$$

We have

$$h\omega(t, \|x\|)\|x\| + O(h)\|x\| \geq \|x + hf(t, x)\|\|x\| - \|x\|^2$$

(by the preceding inequality)

$$\geq (x + hf(t, x), x)_- - \|x\|^2$$

(by Cauchy-Schwartz inequality)

$$\geq \|x\|^2 + (hf(t, x), x)_- - \|x\|^2 \quad (\text{by } (*))$$

$$= h(f(t, x), x)_- .$$

Dividing both members of this inequality by $h > 0$ and taking $\lim_{h\downarrow 0}$ we get $(1)^*$.

$(2) \Rightarrow (2)^*$ and $(3) \Rightarrow (3)^*$ can be proved as $(1) \Rightarrow (1)^*$. The other part of the Theorem will be proved by exhibiting counterexamples in the real line.

$(1)^* \not\Rightarrow (1)$. Let $X = \mathbf{R}$ and define $f, \omega : \mathbf{R} \times]0, 1] \longrightarrow \mathbf{R}$ by

$$f(t, x) = \sin x^{-1} = \omega(t, x).$$

Then $(1)^*$ holds for f, ω. Assume (1) holds for some function O with $\lim_{h\downarrow 0}(O(h)/h) = 0$ and argue for a contradiction. Fix $h \in]0, 1]$. There is a sequence $(x_{h,n})_n$ in $]0, h]$ such that $\lim_n x_{h,n} = 0$ and $\sin(1/x_{h,n}) = -1$. We have

$$|x_{h,n} + hf(t, x_{h,n})| = h - x_{h,n},$$

$$|x_{h,n}| + h\omega(t, |x_{h,n}|) + O(h) = x_{h,n} - h + O(h).$$

Therefore from the assumed (1) we have

$$h - x_{h,n} \leq x_{h,n} - h + O(h).$$

Taking $\lim_n$ we get $h \leq -h + O(h)$ which implies $O(h)/h \geq 2$ (all h), a contradiction.

$(2)^* \not\Rightarrow (2)$. Let $X = \mathbf{R}$ and define $f: \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}$, $\omega: \mathbf{R}^2 \to \mathbf{R}$ by

$$f(t, x) = -tx, \qquad \omega(t, x) = 0.$$

Since $f(t, \cdot)$ is decreasing, $(2)^*$ holds for f, ω. Assume (2) holds for some function O with $\lim_{h\downarrow 0}(O(h)/h) = 0$, and argue for a contradiction. Fix $h > 0$. There are sequences $(x_{h,n})_n$, $(y_{h,n})_n$ of real numbers such that

$$0 < x_{h,n} - y_{h,n} < h \quad \text{and} \quad \lim_n (x_{h,n} - y_{h,n}) = 0.$$

Let $t_{h,n} = 1/(x_{h,n} - y_{h,n})$. We have

$$|x_{h,n} - y_{h,n} + h(f(t_{h,n}, x_{h,n}) - f(t_{h,n}, y_{h,n}))| = h - (x_{h,n} - y_{h,n}),$$

$$|x_{h,n} - y_{h,n}| + h\omega(t_{h,n}, |x_{h,n} - y_{h,n}|) + O(h) = x_{h,n} - y_{h,n} + O(h).$$

Therefore from the assumed (2) we have

$$h - (x_{h,n} - y_{h,n}) \leq x_{h,n} - y_{h,n} + O(h).$$

Taking $\lim_n$ we get $h \leq O(h)$ which implies $O(h)/h \geq 1$ (all h), a contradiction.

$(3)^* \not\Rightarrow (3)$. This follows by taking $g = f$ in the above example used to show $(2)^* \not\Rightarrow (2)$. q.e.d.

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DEPARTMENTO DE MATEMATICA, UNIVERSIDADE DE BRASILIA, 70.000 BRASILIA D.F., BRASIL