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S-TRANSVERSALITY[1]

L. A. FAVARO

ABSTRACT. This paper will extend Thom's transversality theorem to differentiable mappings between foliated manifolds, and deal with mappings with "nice" nontransversal points.

Introduction. Let M and N be differentiable manifolds of dimensions m and n, respectively, and let J and L be foliations on M and N of codimensions d and q, respectively. Let $T(J, L)$ be the set of differentiable maps $f: M \to N$ that carries the leaves of J transversally to the leaves of L. It is easy to see that this set is open and nondense in $C^2(M, N)$ with the C^2-fine topology; see Proposition 1.5 and Corollary 1.7. Our main purpose is to enlarge $T(J, L)$ to obtain a dense set in $C^s(M, N)$ with the C^s-fine topology, for a suitable s, such that the new maps added have only "nice" nontransversal points, i.e. the set of the nontransversal points is a stratified set in the sense of Whitney [2, p. 133]; such a set we will call a manifold collection. With this objective in mind, we introduce the S-transversality maps and prove the more general theorem: the set of these maps is dense in $C^s(M, N)$ with the C^s-fine topology.

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1. Preliminaries. Let M and N be C^r-manifolds of dimension m and n, respectively, $r \geq 1$. Let $f: M \to N$ be a C^r-map, $J^r f(p)$ be the r-jet of f at p and $J^r(M, N)$ the set of r-jets of C^r-maps from M to N. We will denote by $J^r(m, n)$ the set of r-jets at $(0, \ldots, 0)$ of C^r-maps from R^m to R^n that carries the origin to the origin, and by $L^r(m, n) = L^r(m) \times L^r(n)$, where $L^r(m)$ is the group of the invertible elements of $J^r(m, m)$. Let $\pi: J^r(M, N) \to M \times N$ be the map given by $\pi(J^r f(p)) = (p, f(p))$; we may give to $J^r(M, N)$ a differentiable structure such that π turns a differentiable fibration, with fiber $J^r(m, n)$ and structural group $L^r(m, n)$. If S is a submanifold of $J^r(m, n)$

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invariant under the action of $L^r(m, n)$, then S is called a manifold of r-singularities. In this case we will denote by $S(M, N)$ the subbundle of $J^r(M, N)$ that has fiber S. For details see [1].

Now we will give some facts concerning the C^r-fine topology on $C^r(M, N)$; for details see [3] and [4]. For each open subset U of $J^r(M, N)$ let $F(U) = \{f \in C^r(M, N) | J^r f(M) \subset U\}$. The C^r-fine topology on $C^r(M, N)$ has as a base of open sets the sets $F(U)$.

Let A be an open subset of R^m and $h: A \longrightarrow R^n$ be a C^r-map; we write

$$\|h\|_{r,x} = |h_1(x)| + \cdots + |h_n(x)| + \sum_{i=1} |\partial^t h_i(x)|,$$

where $t = (t_1, \ldots, t_m)$ is a sequence of nonnegative integers; we will denote $\|h\|_{r,D} = \sup[\|h\|_{r,x} | x \in D]$ for D a subset of A. Also we will denote $|t| = t_1 + t_2 + \cdots + t_m$.

Given any open covering $(O_i)_{i \in I}$ of N and a C^r-map $f: M \longrightarrow N$, there exist a numerable locally finite refinement $(V_j)_{j=1,2,\ldots}$ of $(f^{-1}(O_i))_{i \in I}$ and a numerable locally finite refinement $(W_k)_{k=1,2,\ldots}$ of $(V_j)_{j=1,2,\ldots}$ such that the V_j are open coordinates sets, $\overline{V}_j$ is compact, $F(\overline{V}_j) \subset O_{i(j)}$ and $\overline{W}_j \subset V_j$. If $a_j: V_j \longrightarrow R^m$ and $b_i: O_i \longrightarrow R^n$ are the coordinate maps, and $\epsilon: M \longrightarrow R_{++}$ is a continuous map, we write

$$N_r(f, V, O, W, \epsilon) = \{g: M \longrightarrow N | \|b_i f a_j^{-1} - b_i g a_j^{-1}\|_{r, a(x)} < \epsilon(x),\ x \in \overline{W}_j,$$
$$\text{and } g(\overline{V}_j) \subset O_{i(j)}, j = 1, 2, \ldots\}.$$

The family of the $N_r(f, V, O, W, \epsilon)$, when ϵ runs on the set of continuous maps from M to R_{++}, is a fundamental neighborhoods system of f in the C^r-fine topology. When V and O are coordinate open sets in M and N, respectively, and $f: M \longrightarrow N$ is a C^r-map such that $f(\overline{V}) \subset O$, then a fundamental neighborhoods system of $f|_V$ is given by the family of sets

$$N^r(f|_V, \epsilon) = \{h: V \longrightarrow N | \|bfa^{-1} - bha^{-1}\|_{r, a(x)} < \epsilon(x)\}.$$

Definition 1.1. A C^r-foliation L on N of codimension $q \leq n$ is given by

(a) $\{U_i\}_{i \in J}$ an open covering of N;

(b) $\{\psi_i\}_{i \in J}$, where $\psi_i: U_i \longrightarrow R^q$ is a C^r-submersion;

(c) for each $x \in U_i \cap U_j$ exists a C^r-diffeomorphism γ_{ji} from an open neighborhood of $\psi_i(x)$ over an open neighbrohood of $\psi_j(x)$, such that $\psi_j = \gamma_{ji} \circ \psi_i$.

The maps $\psi_i: U_i \longrightarrow R^q$ are called a local representation of the foliation L. The leaves of L are the connected submanifolds of M given locally by $\psi_i^{-1}(z)$, $z \in \psi_i(U_i)$. We may choose a local representation of L such that ψ_i is the natural projection $\psi_i(y_1, \ldots, y_n) = (y_1, \ldots, y_q)$.

Definition 1.2. A C^r-map $f: M \longrightarrow N$ is transverse to L at $x \in M$ if $f_x(M_x) + L_{f(x)} = N_{f(x)}$, where f_x is the differential of f at x, M_x is the tangent space to M at x and $L_{f(x)}$ is the tangent space at $f(x)$ to the leaf of L that contains $f(x)$. If J is a C^r-foliation of codimension d on M, then we say that $f\colon M \to N$ is transverse to the couple (J, L) at x if $f_x(J_x) + L_{f(x)} = N_{f(x)}$. When this is true for all $x \in K$, we say that f is transverse to the couple (J, L) on K; we will denote $T(J, L, K) = \{f\colon M \to N \mid f \text{ is transverse to } (J, L) \text{ on } K\}$.

Proposition 1.3. *Let $f\colon M \to N$ be a C^r-map and let J and L be a foliations on M and N of codimensions d and q, respectively, $m \geq d + q$. Then f is transverse to (J, L) at x if and only if $F = (\phi, \psi \circ f|_V)$ is a regular map, where V is an open coordinate set containing x, $\phi\colon V \to R^d$ is a local representation of J, and U is an open coordinate set containing $f(x)$, such that $f(V) \subset U$, and $\psi\colon U \to R^q$ is a local representation of L.*

The proof is very easy.

Now we will show that the set $T(J, L, K)$ is open but not dense in $C^r(M, N)$ with the C^r-fine topology, where $K \subset M$ is a closed subset. Note that $T(J, K, L) = \varnothing$ if $m < d + q$; then we only need prove for $m \geq d + q$.

Lemma 1.4. *With the notations of Proposition 1.3, let W be an open set such that $\overline{W} \subset V$; also suppose that $\overline{V}$ is compact and $K \subset M$ is a closed set. Then the set*

$$T(J, L, V, U, W, K) = \{g\colon M \to N \mid g(\overline{V}) \subset U \text{ and } g \text{ is transverse to } (J, L) \text{ on } \overline{W} \cap K\}$$

is open in the C^r-fine topology, $r \geq 1$.

The proof is standard and we will omit it.

Proposition 1.5. *If $K \subset M$ is closed, then $T(J, L, K)$ is open in $C^r(M, N)$ with the C^r-fine topology, $r \geq 1$.*

Proof. Let $(U_i)_{i \in I}$ be an open covering of N by open coordinate sets, such that the local representation of L on U_i is $\psi(y_1, \dots, y_n) = (y_1, \dots, y_q)$. Given $f \in T(J, L, K)$, let $(W_j)_{j=1,2,\dots}$ and $(V_j)_{j=1,2,\dots}$ be locally finite refinements of the coverings $(f^{-1}(U_i))_{i \in I}$ such that $\overline{V}_j$ is compact and $\overline{W}_j \subset V_j \subset \overline{V}_j \subset f^{-1}(U_{i(j)})$.

By Lemma 1.4, there are $e_j > 0$, $j = 1, 2, \dots$, such that

$$N^j = N_1(f, V_j, U_{i(j)}, \Omega_j, e_j) \subset T(J, L, V_j, U_{i(j)}, W_j, K) = T_j,$$

where the Ω_j are open sets satisfying $\overline{W} \subset \Omega_j \subset \overline{\Omega}_j \subset V_j$, and

$$N_1(f, V_j, U_{i(j)}, \Omega_j, e_j) = \{g\colon M \to N \mid g(\overline{V}_j) \subset U_{i(j)} \text{ and } \|b_i g a_j^{-1} - b_i f a_j^{-1}\|_{1, a_j(x)} < e_j,\ x \in \Omega_j\}.$$

We now take $N_1(f, V, U, \Omega, e)$, where $V = (V_j)$, $U = (U_{i(j)})$, $\Omega = (\Omega_j)$ and

$e = (e_j)$; we have $N_1(f, V, U, \Omega, e) \subset \bigcap_j T_j$. If we note $A(V, U) = \{g \in C^1(M, N) | g(\overline{V}_j) \subset U_{i(j)}\}$, then $A(V, U)$ is open in $C^1(M, N)$ with the C^1-fine topology and $\bigcap_j T_j \subset T(J, L, K) \cap A(V, U)$. But this implies that $N_1(f, V, U, \Omega, e) \subset T(J, L, K)$, and the proof is finished.

Proposition 1.6. *Let J and L be C^2-foliations on M and N of codimensions d and q, respectively; then there is a nonempty open set $\mathfrak{U} \subset C^2(M, N)$ in the C^2-fine topology such that $\mathfrak{U} \cap T(J, L, K) = \Phi$, where $K \subset M$ is a closed set with nonempty interior.*

Proof. If $m < d + q$, we may take $\mathfrak{U} = C^2(M, N)$. Then we will suppose that $m \geq d + q$. Let x be an interior point of K and let $f: M \to N$ be a C^2-map such that x is a singular point of $F = (\phi, \psi \circ f|_V)$ and $J^1F: V \to J^1(V, R^d \times R^q)$ is transverse to $\Sigma(V, R^d \times R^q)$ at x, where $\phi: V \to R^q$ and $\psi: U \to R^q$ are local representations of J and L, respectively, at the neighborhoods V of x and U of $f(x)$, and $\Sigma(V, R^d \times R^q)$ is the singularities set in $J^1(V, R^d \times R^q)$. Also we may suppose $V \subset K$ and the coordinates on V and U such that $\phi(x_1, \ldots, x_m) = (x_1, \ldots, x_d)$ and $\phi(y_1, \ldots, y_n) = (y_1, \ldots, y_q)$.

We know that if $W \subset V$ is an open set and $x \in W$, there is a continuous map $\epsilon: V \to R_{++}$ such that for all C^2-maps $h: V \to R^d \times R^q$ satisfying $h \in N_2(F, \epsilon)$, there is $\overline{x} \in W$, a singular point of h [1, p. 45]. We may take W such that $\overline{W} \subset V$.

Now let Ω be an open set such that $\overline{W} \subset \Omega \subset \overline{\Omega} \subset V$ and let $e: M \to R_{++}$ be a continuous map satisfying $e(x) = \epsilon(x)$ if $x \in \Omega$. Say $A = \{g \in C^2(M, N) | g(\overline{V}) \subset U\}$. By the choice of the coordinates and if $g \in N_2(f, e) \cap A$, we have $(\phi, \psi \circ g|_\Omega) = h|_\Omega$ for some $h \in N_2(F, \epsilon)$. Then $\overline{x}$ is a singular point of $(\phi, \psi \circ g|_V)$, i.e., g is not transverse to (J, L) at $\overline{x}$ and the proof is completed, if we take $\mathfrak{U} = N_2(f, e) \cap A$.

Corollary 1.7. *$T(J, L, M)$ is not dense in $C^2(M, N)$ with the C^2-fine topology.*

S-transversality. Let M and N be s-differentiable manifolds of dimensions m and n, J and L be C^s-foliations dimensions d and q, respectively. Also let $S \subset J^r(m, d + q)$, $r < s$, be an invariant $(s - r)$-differentiable submanifold.

Definition 2.1. We say that $f: M \to N$ is S-transverse to (J, L) at $x \in M$ if $J^r(\phi, \psi \circ f|_V): V \to J^r(V, R^d \times R^q)$ is transverse to $S(V, R^d \times R^q)$ at x, where V and U are open sets containing x and $f(x)$, respectively, with $f(\overline{V}) \subset U$ and $\phi: V \to R^d$ and $\psi: U \to R^q$ the local representations of J and L, respectively. If f is S-transverse to (J, L) at all $x \in K$, $K \subset M$, we say that f is S-transverse to (J, L) on K, and we denote

$$T(J, L, S, K) = \{f \in C^s(M, N) | f \text{ is } S\text{-transverse to } (J, L) \text{ on } K\}.$$

Our main goal is to prove that $T(J, L, S, K)$ is a dense subset of $C^s(M, N)$ with the C^s-fine topology, for a suitable $s \geq r$, when $K \subset M$ is closed. With this objective in mind we define $Z \subset J^r(M, N)$ as the set of $z \in J^r(M, N)$ such that $J^r(\phi, \psi \circ f)(x) \in S(V, R^d \times R^q)$, where $z = J^r f(x)$ and ϕ and ψ are representations of J and L in the neighborhoods V of x and U of $f(x)$, respectively. The definition of Z is independent of the choices of ϕ, ψ and f.

Theorem 2.2. *If $S \subset J^r(m, d+q)$ is an invariant submanifold, then $Z \subset J^r(M, N)$ is void or a submanifold of the same codimension as S and $J^r f$ is transverse to Z at x if and only if f is S-transverse to (J, L) at x.*

Proof. Let $\bar{z} \in Z$ such that $J^r(\phi, \psi \circ f|_V)(\bar{x}) \in S(V, R^d \times R^q)$, where $\bar{z} = J^r f(\bar{x})$ and $\phi: V \to R^d$ and $\psi: U \to R^q$ are representations of J and L, respectively; we may suppose that $f(\bar{V}) \subset U$. If $A(V, U) = \{g: C^s(M, N) | g(\bar{V}) \subset U\}$ and $A^r(V, U) = \{z \in J^r(M, N) | z = J^r g(x), g \in A(V, U), x \in V\}$, then $A^r(V, U)$ is open in $J^r(M, N)$. Define $\theta: A^r(V, U) \to J^r(V, R^d \times R^q)$ by $\theta(z) = J^r(\phi, \psi \circ g|_V)(x)$. We have $Z \cap A^r(V, U) = \theta^{-1}(S(V, R^d \times R^q))$ and the commutative diagram

$$\begin{tikzcd}
A^r(V, U) \arrow[rr, "\theta"] & & J^r(V, R^d \times R^q) \\
& V \arrow[ul, "J^r g"] \arrow[ur, "J^r(\phi, \psi \circ g|_V)"'] &
\end{tikzcd}$$

Then the theorem will follow if we prove that θ is transverse to $S(V, R^d \times R^q)$ at $\bar{z}$. We may choose the coordinates $(x_1, \ldots, x_m)$ in V and $(y_1, \ldots, y_n)$ in U with origins at $\bar{x}$ and $f(\bar{x})$ such that ϕ and ψ are given by $\phi(x_1, \ldots, x_m) = (x_1, \ldots, x_d)$ and $\psi(y_1, \ldots, y_m) = (y_1, \ldots, y_q)$. Then θ is given by

$$\theta\left[x; y; \left(\frac{\partial g^i}{\partial x_j}\right); \left(\frac{\partial^2 g^i}{\partial x_{j_1} \partial x_{j_2}}\right); \ldots; \left(\frac{\partial^r g^i}{\partial x_{j_1} \cdots \partial x_{j_r}}\right)\right]$$

$$= \left[x; x_1, \ldots, x_d, y_1, \ldots, y_q; \begin{pmatrix} I_{d\times d} & 0 \\ \multicolumn{2}{c}{\dfrac{\partial g^k}{\partial x_j}} \end{pmatrix}; \begin{pmatrix} 0 \\ \dfrac{\partial^2 g^k}{\partial x_{j_1} \partial x_{j_2}} \end{pmatrix}; \ldots; \begin{pmatrix} 0 \\ \dfrac{\partial^r g^k}{\partial x_{j_1} \cdots \partial x_{j_r}} \end{pmatrix}\right]$$

where $i = 1, \ldots, n$; $k = 1, \ldots, q$; $j, j_t = 1, \ldots, m$. Note that the $I_{d\times d}$ matrix and the zero matrices that appear above are given by the successive derivatives of $\phi(x_1, \ldots, x_m) = (x_1, \ldots, x_d)$.

From (*) we see that the image of the tangent space to $A^r(V, U)$ by the differential of θ at $\bar{z}$ contains a complementary subspace of the subspace generated by the entries corresponding to the successive derivatives of ϕ. Then to prove that θ is transverse to $S(V, R^d \times R^q)$ at $\bar{z}$, it is enough to prove that the tangent space to $S(V, R^d \times R^q)$ at $\theta(\bar{z})$ projects onto the subspace generated by the entries corresponding to the successive derivatives of ϕ. To do this, take $T\colon V \to V$, a polynomial change of coordinates given by

$$x_i = \sum_{j=1}^{m} a^i_j u_j + \sum_{j_1, j_2 = 1} \frac{1}{2} a^i_{j_1 j_2} x_{j_1} x_{j_2} + \cdots + \sum_{j_1 \cdots j_r} \frac{1}{r} a^i_{j_1 \cdots j_r} x_{j_1} \cdots x_{j_r},$$

$$x_t = u_t, \qquad i = 1, \ldots, d;\ k = d+1, \ldots, m; \quad a^i_{j_1 \cdots j_k} = a^i_{\sigma(j_1) \cdots \sigma(j_k)},$$

where σ is a permutation of $j_1, \ldots, j_k$.

By the chain rule it is possible to see that the element of $L^r(m, d+q)$ corresponding to T takes $J^r(\phi, \psi \circ f)(\bar{x})$ to

$$\left[\bar{u};\ (\phi, \psi \circ f)T(\bar{u});\ \begin{pmatrix} a^i_j \\ \dfrac{\partial g^k}{\partial u_j} \end{pmatrix};\ \begin{pmatrix} a^i_{j_1 j_2} \\ \dfrac{\partial^2 g^k}{\partial u_{j_1} \partial u_{j_2}} \end{pmatrix};\ \begin{pmatrix} a^i_{j_1 \cdots j_r} \\ \dfrac{\partial^r g^k}{\partial u_{j_1} \cdots \partial u_{j_r}} \end{pmatrix}\right],$$

where $T(\bar{u}) = \bar{x}$. Since S is invariant under the action of $L^r(m, d+q)$, we have (**) belonging to $S(V, R^d \times R^q)$ for (a^i_j) near the $(I_{d \times d}\ 0)$ matrix and $a^i_{j_1 \cdots j_k}$ near zero. But this means that $S(V, R^d \times R^q)$ contains small segments in the directions of the entries correspondent to the successive derivatives of ϕ near $\theta(\bar{z})$; then the projection of the tangent space to $S(V, R^d \times R^q)$ at $\theta(\bar{z})$ is onto the subspace generated by the entries correspondent to the successive derivatives of ϕ, and the theorem is proved.

Corollary. *If $K \subset M$ is closed, then $T(J, L, S, K)$ is dense in $C^s(M, N)$ with the C^s-fine topology, $s - r > \max(m - \operatorname{cod} S, 0)$.*

Proof. By Thom's transversality theorem [1, p. 32], the set of maps transverse to Z on K is dense in $C^s(M, N)$ with the C^s-fine topology. Then the corollary follows from the preceding theorem.

3. Geometric interpretation. Let $S(J, L, f) = \{x \in M \mid f$ is not transverse to (J, L) at $x\}$; we are interested in asking for nice properties for this set, as the singularities theory ask for the singularities set $S(f)$.

As we did in the introduction, a manifold-collection will mean a stratified set in the sense of Whitney [2, p. 133].

Proposition 3.1. *If J and L are foliations on M and N of codimension d and q, respectively, $\Sigma \subset J^1(m, d+q)$ is the set of 1-singularities, and $f: M \to N$ is Σ-transverse to (J, L) on M, then*

(a) *$S(J, L, f) = M$ if $m < d + q$;*

(b) *$S(J, L, f) = \emptyset$ or a manifold-collection of codimension $m - (d+q) + 1$, if $m \geq d + q$.*

Proof. Suppose $m \geq d + q$ and say

$$S_k(J, L, f) = \{x \in M \mid \dim[f_x(\Gamma_x) + L_{f(x)}] = n - k\};$$

then $S(J, L, f) = S_1(J, L, f) \cup \cdots \cup S_q(J, L, f)$. Also, if $\phi: V \to R^d$ and $\psi: U \to R^q$ are local representations of J and L, respectively, and $F = (\phi, \psi \circ f|_V)$, we have $S(J, L, f) \cap V = S(F)$ and $S_k(J, L, f) \cap V = S_k(F)$. Since J^1F is transverse to $\Sigma(V, R^d \times R^q)$, then $S(F) = S_1(F) \cup \cdots \cup S_q(F)$ is a manifold-collection and so is $S(J, L, f)$.

Corollary 3.2. *The set of C^s-maps $f: M \to N$ such that $S(J, L, f)$ is a manifold-collection is a dense subset of $C^s(M, N)$ with the C^s-fine topology, $s - 1 > \max(m - \operatorname{cod} \Sigma, 0)$.*

Let $\Sigma_{i_1, \ldots, i_r}$, $i_1 = i_r = 1$, be the singularities set defined in [5]. Let L be a foliation of codimension one on N, and $f: M \to N$ a C^{r+1}-map such that $J^r f$ is transverse to $\Sigma_{i_1, \ldots, i_r}$ on M, and suppose that f is Σ-transverse to L. Using the normal forms of [6], the following was proved in [7]:

(a) if $x \in T(L, f)$, then the tangent space to $f(S(f))$ at $f(x)$ cuts L transversally;

(b) if $x \in S(L, f) \cap \Sigma_{i_1 i_2}(f)$, $i_1 = i_2 = 1$, and $\psi: U \to R$ is a local representation of L near $f(x)$, and V is an open neighborhood of x such that $f(V) \subset U$, then $\psi \circ f$ restricted to $\Sigma_{i_1 i_2}(f) \cap V$ is nondegenerated;

(c) if $r > 2$, the hypotheses are not sufficient to give a definite answer; see the example below.

Example. Let $f: R^4 \to R^4$ given by $f(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_1 x_4 + x_2 x_4^2 + x_4^4)$ and let L be the foliation represented by $(y_1, y_2, y_3, y_4) = y_2 y_3 + y_4$. We have 0 a nondegenerated singular point of $\psi \circ f$ and $\psi \circ f$ restricted to $\Sigma_{111}(f)$ is identically zero. Then this situation may change by small deformations.

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DEPARTMENT OF MATHEMATICS, BRANDEIS UNIVERSITY, WALTHAM, MASSACHUSETTS 02154

Current address: Instituto de Ciências Matemáticas, Universidade de São Paulo, C.P. 359–378, CEP 13560, São Carlos, Brasil