S-TRANSVERSALITY

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ABSTRACT. This paper will extend Thom's transversality theorem to differentiable mappings between foliated manifolds, and deal with mappings with "nice" nontransversal points.

Introduction. Let $M$ and $N$ be differentiable manifolds of dimensions $m$ and $n$, respectively, and let $F$ and $L$ be foliations on $M$ and $N$ of codimensions $d$ and $q$, respectively. Let $T(J, L)$ be the set of differentiable maps $f: M \to N$ that carries the leaves of $F$ transversally to the leaves of $L$. It is easy to see that this set is open and nondense in $C^2(M, N)$ with the $C^2$-fine topology; see Proposition 1.5 and Corollary 1.7. Our main purpose is to enlarge $T(J, L)$ to obtain a dense set in $C^s(M, N)$ with the $C^s$-fine topology, for a suitable $s$, such that the new maps added have only "nice" nontransversal points, i.e., the set of the nontransversal points is a stratified set in the sense of Whitney [2, p. 133]; such a set we will call a manifold collection. With this objective in mind, we introduce the $S$-transversality maps and prove the more general theorem: the set of these maps is dense in $C^s(M, N)$ with the $C^s$-fine topology.

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1. Preliminaries. Let $M$ and $N$ be $C^r$-manifolds of dimension $m$ and $n$, respectively, $r > 1$. Let $f: M \to N$ be a $C^r$-map, $j^r_f(p)$ be the $r$-jet of $f$ at $p$ and $J^r_f(M, N)$ the set of $r$-jets of $C^r$-maps from $M$ to $N$. We will denote by $J^r(m, n)$ the set of $r$-jets at $(0, \ldots, 0)$ of $C^r$-maps from $R^m$ to $R^n$ that carries the origin to the origin, and by $L^r(m, n) = L^r(m) \times L^r(n)$, where $L^r(m)$ is the group of the invertible elements of $J^r(m)$ and $J^r(n)$.

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invariant under the action of \( L'(m, n) \), then \( S \) is called a manifold of \( r \)-singularities. In this case we will denote by \( S(M, N) \) the subbundle of \( J'(M, N) \) that has fiber \( S \). For details see \([1]\).

Now we will give some facts concerning the \( C^r \)-fine topology on \( C'(M, N) \); for details see \([3]\) and \([4]\). For each open subset \( U \) of \( J'(M, N) \) let \( F(U) = \{ f \in C'(M, N) | J'(M) \subset U \} \). The \( C^r \)-fine topology on \( C'(M, N) \) has as a base of open sets the sets \( F(U) \).

Let \( A \) be an open subset of \( R^m \) and \( b: A \rightarrow R^n \) be a \( C^r \)-map; we write
\[
\|f\|_{r, x} = |b_1(x)| + \cdots + |b_n(x)| + \sum_{i=1}^m |\partial^i b_j(x)|,
\]
where \( t = (t_1, \ldots, t_m) \) is a sequence of nonnegative integers; we will denote \( \|h\|_{r,D} = \sup \{\|h\|_{r,x} | x \in D \} \) for \( D \) a subset of \( A \). Also we will denote \( |t| = t_1 + t_2 + \cdots + t_m \).

Given any open covering \( \{O_j\} \) of \( N \) and a \( C^r \)-map \( f: M \rightarrow N \), there exist a numerable locally finite refinement \( \{V_j\} \) of \( \{O_j\} \) and a numerable locally finite refinement \( \{W_j\} \) of \( \{V_j\} \) such that the \( V_j \) are open coordinates sets, \( V_j \) is compact, \( F(V_j) \subset O_{i(j)} \) and \( W_j \subset V_j \). If \( a: V_j \rightarrow R^m \) and \( b: O_i \rightarrow R^n \) are the coordinate maps, and \( \epsilon: M \rightarrow R^+ \) is a continuous map, we write
\[
N(f, V, O, W, \epsilon) = \{ p: M \rightarrow N | \|b|^{-1} - b|^{-1}\|_{r, a(x)} < \epsilon(x), x \in V_j, \text{ and } g(V_j) \subset O_{i(j), j=1,2,\ldots} \}.
\]
The family of the \( N(f, V, O, W, \epsilon) \), when \( \epsilon \) runs on the set of continuous maps from \( M \) to \( R^+ \), is a fundamental neighborhoods system of \( f \) in the \( C^r \)-fine topology. When \( V \) and \( O \) are coordinate open sets in \( M \) and \( N \), respectively, and \( f: M \rightarrow N \) is a \( C^r \)-map such that \( f(V) \subset O \), then a fundamental neighborhoods system of \( f \) in \( V \) is given by the family of sets
\[
N(f|_V, \epsilon) = \{ q: V \rightarrow N | \|b|^{-1} - b|^{-1}\|_{r, a(x)} < \epsilon(x) \}.
\]

**Definition 1.1.** A \( C^r \)-foliation \( L \) on \( N \) of codimension \( q \leq n \) is given by
(a) \( \{U_i\} \) an open covering of \( N \);  
(b) \( \{\psi_i\} \), where \( \psi_i: U_i \rightarrow R^q \) is a \( C^r \)-submersion;  
(c) for each \( x \in U_i \cap U_j \) exists a \( C^r \)-diffeomorphism \( \gamma_{ji} \) from an open neighborhood of \( \psi_i(x) \) over an open neighborhood of \( \psi_j(x) \), such that \( \psi_j = \gamma_{ji} \circ \psi_i \).

The maps \( \psi_i: U_i \rightarrow R^q \) are called a local representation of the foliation \( L \). The leaves of \( L \) are the connected submanifolds of \( M \) given locally by \( \psi_i^{-1}(z) \), \( z \in \psi_i(U_i) \). We may choose a local representation of \( L \) such that \( \psi_i \) is the natural projection \( \psi_i(y_1, \ldots, y_n) = (y_1, \ldots, y_q) \).

**Definition 1.2.** A \( C^r \)-map \( f: M \rightarrow N \) is transverse to \( L \) at \( x \in M \) if \( f_x(M_x) + L_x \cap N = N(f(x)) \), where \( f_x \) is the differential of \( f \) at \( x \), \( M_x \) is the tangent space.
to \( M \) at \( x \) and \( L_{f(x)} \) is the tangent space at \( f(x) \) to the leaf of \( L \) that contains \( f(x) \). If \( J \) is a \( C^r \)-foliation of codimension \( d \) on \( M \), then we say that \( f: M \to N \) is transverse to the couple \((J, L)\) at \( x \) if \( f_x(J_x) + L_{f(x)} = N_{f(x)} \). When this is true for all \( x \in K \), we say that \( f \) is transverse to the couple \((J, L)\) on \( K \); we will denote \( T(f, J, L, K) = \{ f: M \to N \mid f \text{ is transverse to } (J, L) \text{ on } K \} \).

**Proposition 1.3.** Let \( f: M \to N \) be a \( C^r \)-map and let \( J \) and \( L \) be foliations on \( M \) and \( N \) of codimensions \( d \) and \( q \), respectively, \( m \geq d + q \). Then \( f \) is transverse to \((J, L)\) at \( x \) if and only if \( F = (\phi, \psi \circ f|_V) \) is a regular map, where \( V \) is an open coordinate set containing \( x \), \( \phi: V \to R^d \) is a local representation of \( J \), and \( U \) is an open coordinate set containing \( f(x) \), such that \( f(V) \subseteq U \), and \( \psi: U \to R^q \) is a local representation of \( L \).

The proof is very easy.

Now we will show that the set \( T(f, L, K) \) is open but not dense in \( C^r(M, N) \) with the \( C^r \)-fine topology, where \( K \subseteq M \) is a closed subset. Note that \( T(f, K, L) = \emptyset \) if \( m < d + q \); then we only need prove for \( m \geq d + q \).

**Lemma 1.4.** With the notations of Proposition 1.3, let \( W \) be an open set such that \( W \subseteq V \); also suppose that \( \overline{V} \) is compact and \( K \subseteq M \) is a closed set. Then the set

\[
T(f, L, V, U, W, K) = \{ g: M \to N \mid g(V) \subseteq U, \text{ and } g \text{ is transverse to } (J, L) \text{ on } W \cap K \}
\]

is open in the \( C^r \)-fine topology, \( r \geq 1 \).

The proof is standard and we will omit it.

**Proposition 1.5.** If \( K \subseteq M \) is closed, then \( T(f, L, K) \) is open in \( C^r(M, N) \) with the \( C^r \)-fine topology, \( r \geq 1 \).

**Proof.** Let \( \{ U_i \}_{i \in I} \) be an open covering of \( N \) by open coordinate sets, such that the local representation of \( L \) on \( U_i \) is \( \psi(y_1, \ldots, y_n) = (y_1, \ldots, y_q) \). Given \( f \in T(f, J, L, K) \), let \( \{ W_j \}_{j=1,2,\ldots} \) and \( \{ V_j \}_{j=1,2,\ldots} \) be locally finite refinements of the coverings \( \{ f^{-1}(U_i) \}_{i \in I} \) such that \( \overline{V_j} \) is compact and \( \overline{W_j} \subseteq V_j \subseteq f^{-1}(U_{i(j)}) \).

By Lemma 1.4, there are \( e_j > 0 \), \( j = 1, 2, \ldots \), such that

\[
V_j = \bigcap_i \{ f_i(V_j, U_{i(j)}, \Omega_j, e_j) \subseteq T(f, J, L, V_j, U_{i(j)}, W_j, K) = T_{i(j)}
\]

where the \( \Omega_j \) are open sets satisfying \( \overline{W_j} \subseteq \overline{\Omega_j} \subseteq \overline{V_j} \), and

\[
\bigcap_i \{ f_i(V_j, U_{i(j)}, \Omega_j, e_j) = \{ g: M \to N \mid g(V_j) \subseteq U_{i(j)} \}
\]

and \( \| b_ja_j^{-1} - b_i a_i^{-1} \|_{1, a_j(x)} < e_j, \ x \in \Omega_j \). We now take \( N_{i(j)}(f_i(V_j, \Omega_j, \Omega_j, e_j)) \), where \( V_j = (V_j), \ U_{i(j)} = (U_{i(j)}), \Omega = (\Omega_j) \) and
$e = (e, \emptyset)$; we have $N_1(f, V, U, \Omega, e) \subset \bigcap_j T_j$. If we note $A(V, U) = \{ g \in C^1(M, N) | g(V, U) \subset U \}$, then $A(V, U)$ is open in $C^1(M, N)$ with the $C^1$-fine topology and $\bigcap_j T_j \subset T(J, L, K) \cap A(V, U)$. But this implies that $N_1(f, V, U, \Omega, e) \subset T(J, L, K)$, and the proof is finished.

**Proposition 1.6.** Let $J$ and $L$ be $C^2$-foliations on $M$ and $N$ of codimensions $d$ and $q$, respectively; then there is a nonempty open set $\mathcal{U} \subset C^2(M, N)$ in the $C^2$-fine topology such that $\mathcal{U} \cap T(J, L, K) = \emptyset$, where $K \subset M$ is a closed set with nonempty interior.

**Proof.** If $m < d + q$, we may take $\mathcal{U} = C(M, N)$. Then we will suppose that $m \geq d + q$. Let $x$ be an interior point of $K$ and let $f: M \to N$ be a $C^2$-map such that $x$ is a singular point of $F = (\phi, \psi)_{|W}$ and $J^1 F: V \to J^1(V, R^d \times R^q)$ is transverse to $\Sigma(V, R^d \times R^q)$ at $x$, where $\phi: V \to R^q$ and $\psi: U \to R^q$ are local representations of $J$ and $L$, respectively, at the neighborhoods $V$ of $x$ and $U$ of $f(x)$, and $\Sigma(V, R^d \times R^q)$ is the singularities set in $J^1(V, R^d \times R^q)$. Also we may suppose $V \subset K$ and the coordinates on $V$ and $U$ such that $\phi(x_1, \ldots, x_m) = (x_1, \ldots, x_d)$ and $\phi(y_1, \ldots, y_n) = (y_1, \ldots, y_q)$.

We know that if $W \subset V$ is an open set and $x \in W$, there is a continuous map $e: V \to R^{++}$ such that for all $C^2$-maps $h: V \to R^d \times R^q$ satisfying $h(\in A)$, we have $(\phi, \psi \circ h|_W) = h|_W$ for some $h \in N_2(F, e)$. We may take $W$ in such that $W \subset V$.

Now let $\Omega$ be an open set such that $\overline{W} \subset \Omega \subset \Omega \subset V$ and let $e: M \to R^{++}$ be a continuous map satisfying $e(x) = e(x)$ if $x \in \Omega$. Say $A = \{ g \in C^2(M, N) | g(\overline{V}) \subset U_1 \}$. By the choice of the coordinates and if $g \in N_2(f, e) \cap A$, we have $(\phi, \psi \circ g|_\Omega) = h|_\Omega$ for some $h \in N_2(F, e)$. Then $\overline{x}$ is a singular point of $(\phi, \psi \circ g|_V)$, i.e., $g$ is not transverse to $(J, L)$ at $\overline{x}$ and the proof is completed, if we take $\mathcal{U} = N_2(f, e) \cap A$.

**Corollary 1.7.** $T(J, L, M)$ is not dense in $C^2(M, N)$ with the $C^2$-fine topology.

**S-transversality.** Let $M$ and $N$ be $s$-differentiable manifolds of dimensions $m$ and $n$, $J$ and $L$ be $C^s$-foliations dimensions $d$ and $q$, respectively. Also let $S \subset J^s(m, d + q)$, $r \subset s$, be an invariant $(s - r)$-differentiable sub-manifold.

**Definition 2.1.** We say that $f: M \to N$ is $S$-transverse to $(J, L)$ at $x \in M$ if $J^r(\phi, \psi \circ f|_V): V \to J^r(V, R^d \times R^q)$ is transverse to $S(V, R^d \times R^q)$ at $x$, where $V$ and $U$ are open sets containing $x$ and $f(x)$, respectively, with $f(\overline{V}) \subset U$ and $\phi: V \to R^d$ and $\psi: U \to R^q$ the local representations of $J$ and $L$, respectively. If $f$ is $S$-transverse to $(J, L)$ at all $x \in K$, $K \subset M$, we say that $f$ is $S$-transverse to $(J, L)$ on $K$, and we denote

$S(f, g, K) = \{ f \in C(M, N) | f \text{ is } S \text{-transverse to } (J, L) \text{ on } K \}$. 

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Our main goal is to prove that \( T(J, L, S, K) \) is a dense subset of \( C^s(M, N) \) with the \( C^s \)-fine topology, for a suitable \( s \geq r \), when \( K \subset M \) is closed. With this objective in mind we define \( Z \subset J^r(M, N) \) as the set of \( z \in J^r(M, N) \) such that \( J^r(\phi, \psi \circ j)(x) \in S(V, R^d \times R^q) \), where \( z = J^r(x) \) and \( \phi \) and \( \psi \) are representations of \( J \) and \( L \) in the neighborhoods \( V \) of \( x \) and \( U \) of \( f(x) \), respectively. The definition of \( Z \) is independent of the choices of \( \phi, \psi \) and \( f \).

**Theorem 2.2.** If \( S \subset J^r(m, d + q) \) is an invariant submanifold, then \( Z \subset J^r(M, N) \) is void or a submanifold of the same codimension as \( S \) and \( J^r \) is transverse to \( Z \) at \( x \) if and only if \( f \) is \( S \)-transverse to \( (J, L) \) at \( x \).

**Proof.** Let \( \overline{x} \in Z \) such that \( J^r(\phi, \psi \circ f|_{V})(\overline{x}) \in S(V, R^d \times R^q) \), where \( \overline{x} = J^r(\overline{x}) \) and \( \phi: V \to R^d \) and \( \psi: U \to R^q \) are representations of \( J \) and \( L \), respectively; we may suppose that \( f(V) \subset U \). If \( A(V, U) = \{g: C^s(M, N)|g(V) \subset U\} \) and \( A'(V, U) = \{z \in J^r(M, N)|z = J^r(x), g \in A(V, U), x \in V\} \), then \( A'(V, U) \) is open in \( J^r(M, N) \). Define \( \theta: A'(V, U) \to J^r(V, R^d \times R^q) \) by \( \theta(z) = J^r(\phi, \psi \circ g|_{V})(x) \). We have \( Z \cap A'(V, U) = \theta^{-1}(S(V, R^d \times R^q)) \) and the commutative diagram

\[
\begin{array}{ccc}
A'(V, U) & \xrightarrow{\theta} & J^r(V, R^d \times R^q) \\
\downarrow & & \downarrow \theta \\
J^r(\phi, \psi \circ g|_{V}) & & J^r(\phi, \psi \circ g|_{V})
\end{array}
\]

Then the theorem will follow if we prove that \( \theta \) is transverse to \( S(V, R^d \times R^q) \) at \( \overline{x} \). We may choose the coordinates \( (x_1, \ldots, x_m) \) in \( V \) and \( (y_1, \ldots, y_q) \) in \( U \) with origins at \( \overline{x} \) and \( f(\overline{x}) \) such that \( \phi \) and \( \psi \) are given by \( \phi(x_1, \ldots, x_m) = (x_1, \ldots, x_d) \) and \( \psi(y_1, \ldots, y_m) = (y_1, \ldots, y_q) \). Then \( \theta \) is given by

\[
\theta = \begin{bmatrix}
x; y; \left(\frac{\partial g_{ij}}{\partial x_j}\right); \left(\frac{\partial^2 g_{ij}}{\partial x_{i1} \partial x_{j2}}\right); \ldots; \left(\frac{\partial^r g_{ij}}{\partial x_{i1} \ldots \partial x_{ir}}\right)
\end{bmatrix}
\]

where \( i = 1, \ldots, m \), \( k = 1, \ldots, q \), \( r = 1, \ldots, m \). Note that the \( I_{d \times d} \) matrix
and the zero matrices that appear above are given by the successive derivatives of \( \phi(x_1, \ldots, x_m) = (x_1, \ldots, x_d) \).

From (*) we see that the image of the tangent space to \( \Lambda^r(V, U) \) by the differential of \( \theta \) at \( \overline{z} \) contains a complementary subspace of the subspace generated by the entries corresponding to the successive derivatives of \( \phi \).

Then to prove that \( \theta \) is transverse to \( S(V, R^d \times R^q) \) at \( \overline{z} \), it is enough to prove that the tangent space to \( S(V, R^d \times R^q) \) at \( O(\overline{z}) \) projects onto the subspace generated by the entries corresponding to the successive derivatives of \( \phi \). To do this, take \( T: V \to V \), a polynomial change of coordinates given by

\[
x_i = \sum_{j=1}^{m} a^i_{j} u_j + \sum_{j_1, j_2=1}^{m} \frac{1}{2} a^{i}_{j_1 j_2} x_{j_1} x_{j_2} + \cdots + \sum_{j_1 \cdots j_r} \frac{1}{r!} a^{i}_{j_1 \cdots j_r} x_{j_1} \cdots x_{j_r},
\]

\[
x_i = u_i, \quad i = 1, \ldots, d; \quad k = d + 1, \ldots, m; \quad a^{i}_{j_1 \cdots j_k} = a^{i}_{\sigma(j_1) \cdots \sigma(j_k)},
\]

where \( \sigma \) is a permutation of \( j_1, \ldots, j_k \).

By the chain rule it is possible to see that the element of \( L^r(m, d + q) \) corresponding to \( T \) takes \( J^r(\phi, \psi \circ f)(\overline{z}) \) to

\[
\begin{bmatrix}
\overline{u} ; (\phi, \psi \circ f) T(\overline{u}) ; \left( \frac{\partial a_i^j}{\partial u_j} \right) ; \left( \frac{\partial^2 a_i^j}{\partial u_{j_1} \partial u_{j_2}} \right) ; \left( \frac{\partial^r a_i^j}{\partial u_{j_1} \cdots \partial u_{j_r}} \right)
\end{bmatrix},
\]

where \( T(\overline{u}) = \overline{z} \). Since \( S \) is invariant under the action of \( L^r(m, d + q) \), we have (**) belonging to \( S(V, R^d \times R^q) \) for \( (a^i_j) \) near the \((d \times d) 0\) matrix and \( a^{i}_{j_1 \cdots j_k} \) near zero. But this means that \( S(V, R^d \times R^q) \) contains small segments in the directions of the entries correspondent to the successive derivatives of \( \phi \) near \( \theta(\overline{z}) \); then the projection of the tangent space to \( S(V, R^d \times R^q) \) at \( \theta(\overline{z}) \) is onto the subspace generated by the entries correspondent to the successive derivatives of \( \phi \), and the theorem is proved.

**Corollary.** If \( K \subset M \) is closed, then \( T(\phi, L, S, K) \) is dense in \( C^s(M, N) \) with the \( C^s \)-fine topology, \( s - r > \max(m - \text{cod } S, 0) \).

**Proof.** By Thom's transversality theorem [1, p. 32], the set of maps transverse to \( Z \) on \( K \) is dense in \( C^s(M, N) \) with the \( C^s \)-fine topology.

Then the corollary follows from the preceding theorem.

3. **Geometric interpretation.** Let \( S(\phi, L, f) = \{ x \in M \mid f \text{ is not transverse to } (\phi, L) \text{ at } x \} \); we are interested in asking for nice properties for this set, as the singularities theory ask for the singularities set \( S(\phi) \).

As we did in the introduction, a manifold-collection will mean a stratified set in the sense of Whitney [2, p. 133].
Proposition 3.1. If \( J \) and \( L \) are foliations on \( M \) and \( N \) of codimension \( d \) and \( q \), respectively, \( \Sigma \subset J^1(m, d + q) \) is the set of \( 1 \)-singularities, and \( f: M \to N \) is \( \Sigma \)-transverse to \((J, L)\) on \( M \), then

(a) \( S(J, L, f) = M \) if \( m < d + q \);

(b) \( S(J, L, f) = \emptyset \) or a manifold-collection of codimension \( m - (d + q) + 1 \), if \( m \geq d + q \).

Proof. Suppose \( m \geq d + q \) and say

\[
S_k(J, L, f) = \{ x \in M | \dim \left[ x \Gamma_x + Lf(x) \right] = n - k \};
\]

then \( S(J, L, f) = \bigcup S_k(J, L, f) \). Also, if \( \phi: V \to \mathbb{R}^d \) and \( \psi: U \to \mathbb{R}^q \) are local representations of \( J \) and \( L \), respectively, and \( F = (\phi, \psi \circ f^{-1}) \), we have \( S(J, L, f) \cap V = S(F) \) and \( S_k(J, L, f) \cap V = S_k(F) \).

Since \( J^1F \) is transverse to \( \Sigma(V, \mathbb{R}^d \times \mathbb{R}^q) \), then \( S(F) = \bigcup S_k(F) \) is a manifold-collection and so is \( S(J, L, f) \).

Corollary 3.2. The set of \( C^s \)-maps \( f: M \to N \) such that \( S(J, L, f) \) is a manifold-collection is a dense subset of \( C^s(M, N) \) with the \( C^s \)-fine topology, \( s - 1 > \max(m - \text{cod} \Sigma, 0) \).

Let \( \Sigma_{i_1, \ldots, i_r} i_1 = i_r = 1 \), be the singularities set defined in [5]. Let \( L \) be a foliation of codimension one on \( N \), and let \( \Sigma_{i_1, \ldots, i_r} \) be a \( C^{r+1} \)-map such that \( f^\prime \) is transverse to \( \Sigma_{i_1, \ldots, i_r} \) on \( M \), and suppose that \( f \) is \( \Sigma \)-transverse to \( L \). Using the normal forms of [6], the following was proved in [7]:

(a) if \( x \in T(L, f) \), then the tangent space to \( f(S(f)) \) at \( f(x) \) cuts \( L \) transversely;

(b) if \( x \in S(L, f) \cap \Sigma_{i_1 i_2} (f), i_1 = i_2 = 1 \), and \( \psi: U \to \mathbb{R} \) is a local representation of \( L \) near \( f(x) \), and \( V \) is an open neighborhood of \( x \) such that \( f(V) \subset U \), then \( \psi \circ f \) restricted to \( \Sigma_{i_1 i_2} (f) \cap V \) is nondegenerated;

(c) if \( r > 2 \), the hypotheses are not sufficient to give a definite answer; see the example below.

Example. Let \( f: R^4 \to R^4 \) given by \( f(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_1 x_4 + x_2 x_3 + x_4) \) and let \( L \) be the foliation represented by \( \langle y_1, y_2, y_3, y_4 \rangle = y_2 y_3 + y_4 \). We have \( 0 \) a nondegenerate singular point of \( \psi \circ f \) and \( \psi \circ f \) restricted to \( \Sigma_{i_1 i_2} (f) \) is identically zero. Then this situation may change by small deformations.

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