

SUPPORT PROPERTIES OF GAUSSIAN PROCESSES OVER SCHWARTZ SPACE¹

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ABSTRACT. We utilize the concept of an abstract Wiener space to prove a converse to a theorem of Minlos, thereby obtaining necessary and sufficient conditions for a Hilbert subspace of $\mathcal{S}'(\mathbb{R}^d)$ to support a given Gaussian process over $\mathcal{S}(\mathbb{R}^d)$.

Stochastic processes over $\mathcal{S}(\mathbb{R}^d)$ are of current interest as elements in the construction of relativistic Boson field theories (see Nelson [6], [7]). The basic process corresponding to the free Euclidean field of mass m is the Gaussian process over \mathcal{S} of mean 0 and covariance $(g, (-\Delta + m^2)^{-1}f)_{L^2(\mathbb{R}^d)}$. According to a theorem of Minlos [4], this process may be realized on \mathcal{S}' , the topological dual of \mathcal{S} , by $\varphi(f): q \rightarrow \langle f, q \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the \mathcal{S} - \mathcal{S}' pairing. That is, there is a Borel measure μ on \mathcal{S}' so that φ maps \mathcal{S} to Gaussian random variables over (\mathcal{S}', μ) with mean 0 and above specified covariances.

We will say that μ is supported on a Hilbert space \mathcal{H} if $\mathcal{H} \subset \mathcal{S}'$, the injection of \mathcal{H} into \mathcal{S}' is continuous, and there is a Borel measure μ_0 on \mathcal{H} so that the restriction of $\varphi(f)$ to \mathcal{H} realizes on (\mathcal{H}, μ_0) the Gaussian process over \mathcal{S} of mean 0 and specified covariance. Support properties of μ have been studied by Reed and Rosen [8]. They utilized a theorem of Minlos to show that certain \mathcal{H} 's support μ , and they showed by a rather lengthy direct computation that others failed to support. Related results concerning the support of μ may be found in the recent work of Cannon [1] and of Colella and Lanford [2]. In this note we show that Minlos' sufficient condition for μ to be supported on \mathcal{H} is, in fact, necessary.

Proposition. *Let $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ be continuous inner products on \mathcal{S} such that $\|f\|_2 \leq c\|f\|_1$ for some constant c and for all f in \mathcal{S} . Let \mathcal{H}_1 and \mathcal{H}_2 be the Hilbert space completions of \mathcal{S} with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Since \mathcal{S} is separable, each \mathcal{H}_i is also separable. Let (φ, μ) be the realization on \mathcal{S}' of the Gaussian process over \mathcal{S} of mean 0 and covariance $(g, f)_2$. Then \mathcal{H}'_1 supports μ if and only if the natural injection $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ is Hilbert-Schmidt.*

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Remark 1. The following are equivalent definitions of the natural injection $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ being \mathcal{H} - \mathcal{S} (Hilbert-Schmidt): (i) (used by Gelfand-Vilenkin [4]) There exists a positive symmetric \mathcal{H} - \mathcal{S} operator A_1 on \mathcal{H}_1 such that $(f, g)_2 = (Af, Ag)_1$. (ii) (used by Reed-Rosen [8]) There exists a 1-1 \mathcal{H} - \mathcal{S} operator A_2 on \mathcal{H}_2 such that $\mathcal{S} \subset A_2\mathcal{H}_2$ and \mathcal{H}_1 is the set $A_2\mathcal{H}_2$ with norm $\|f\|_1 = \|A_2^{-1}f\|_2$. (iii) There exists a positive symmetric \mathcal{H} - \mathcal{S} operator A_3 on \mathcal{H}'_2 such that $(q, p)_{1'} = (A_3q, A_3p)_{2'}$.

Remark 2. \mathcal{S} may be replaced by a nuclear space \mathfrak{E} .

Proof of Proposition. We make the identifications by injection $\mathcal{S} \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2$, by restriction $\mathcal{H}'_2 \subseteq \mathcal{H}'_1 \subseteq \mathcal{S}'$ and also canonically identify \mathcal{H}_i with \mathcal{H}''_i ($i = 1, 2$). The map $\varphi(f)q \rightarrow \langle f, q \rangle$ furnishes a densely defined linear mapping of \mathcal{H}''_1 or \mathcal{H}''_2 to Gaussian random variables of mean 0 and covariance specified by the \mathcal{H}''_2 inner product. Since the \mathcal{H}''_2 inner product is continuous on \mathcal{H}''_1 , we obtain a weak distribution [5] over \mathcal{H}'_1 which is the restriction to \mathcal{H}''_1 of the unit normal distribution over \mathcal{H}'_2 . This weak distribution uniquely determines a (finitely additive) cylinder set measure μ_1 on \mathcal{H}'_1 .

\mathcal{H}'_1 supports μ if and only if μ_1 is countably additive on the ring of cylinder sets in \mathcal{H}'_1 . The separability of the Hilbert spaces allows us to apply a theorem of Dudley, Feldman and LeCam [3], which asserts that countable additivity of μ_1 is equivalent to the pair $(\mathcal{H}'_2, \mathcal{H}'_1)$ forming an abstract Wiener space in the sense of L. Gross [5]. It is well known (and very easy to calculate) that if \mathcal{K}_1 and \mathcal{K}_2 are two real separable Hilbert spaces, then $(\mathcal{K}_1, \mathcal{K}_2)$ forms an abstract Wiener space if \mathcal{K}_2 is the completion of \mathcal{K}_1 with respect to an inner product $(f, g)_2 = (Af, Ag)_1$, where A is positive \mathcal{H} - \mathcal{S} on \mathcal{K}_1 .

Conversely, we claim that if a real separable Hilbert space \mathcal{K}_2 arises as the completion of a Hilbert space \mathcal{K}_1 with respect to a continuous norm $\|\cdot\|_2$ on \mathcal{K}_1 , and if the pair $(\mathcal{K}_1, \mathcal{K}_2)$ forms an abstract Wiener space, then $(f, g)_2 = (Af, Ag)_1$ where A is positive, symmetric and \mathcal{H} - \mathcal{S} on \mathcal{K}_1 . Let us make the identifications $\mathcal{K}'_2 \subseteq \mathcal{K}'_1 \approx \mathcal{K}_1 \subseteq \mathcal{K}_2$, the first containment by restriction and the second by the canonical injection. Then it follows from [5, Corollary 5] that the canonical isomorphism M of \mathcal{K}_2 onto \mathcal{K}'_2 has the property that when restricted to \mathcal{K}_1 and viewed as an operator M_1 on \mathcal{K}_1 , it is positive symmetric and of trace class. But this means that $(f, g)_2 = (f, M_1g)_1 = (\sqrt{M}_1f, \sqrt{M}_1g)_1$, where \sqrt{M}_1 is \mathcal{H} - \mathcal{S} .

REFERENCES

1. J. Cannon, *Continuous sample paths in quantum field theory*, *Comm. Math. Phys.* 35 (1974), 215–233.
2. P. Colella and O. Lanford, *Sample field behavior for the free Markov random field*, *Lecture Notes in Physics*, vol. 25, *Constructive Quantum Field Theory*, Springer-Verlag, Berlin and New York, 1973.

3. R. M. Dudley, J. Feldman and L. Le Cam, *On seminorms and probabilities, and abstract Wiener spaces*, *Ann. of Math.* (2) 93 (1971), 390–408. MR 43 #4995.
4. I. M. Gel'fand and N. Ja. Vilenkin, *Generalized functions*. Vol. 4: *Some applications of harmonic analysis*, Fizmatgiz, Moscow, 1961; English transl., Academic Press, New York, 1964. MR 26 #4173; 30 #4152.
5. L. Gross, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. II: Contributions to Probability Theory, part I, Univ. California Press, Berkeley, Calif., 1967, pp. 31–42. MR 35 #3027.
6. E. Nelson, *Construction of quantum fields from Markoff fields*, *J. Functional Analysis* 12 (1973), 97–112.
7. ———, *Probability theory and Euclidean field theory*, Lecture Notes in Physics, vol. 25, Constructive Quantum Field Theory, Springer-Verlag, Berlin and New York, 1973.
8. M. Reed and L. Rosen, *Support properties of the free measure for Boson fields*, *Comm. Math. Phys.* 36 (1974), 123–132.

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