SUPPORT PROPERTIES OF GAUSSIAN PROCESSES
OVER SCHWARTZ SPACE

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ABSTRACT. We utilize the concept of an abstract Wiener space to prove a converse to a theorem of Minlos, thereby obtaining necessary and sufficient conditions for a Hilbert subspace of \( S'(\mathbb{R}^d) \) to support a given Gaussian process over \( S(\mathbb{R}^d) \).

Stochastic processes over \( S(\mathbb{R}^d) \) are of current interest as elements in the construction of relativistic Boson field theories (see Nelson [6], [7]). The basic process corresponding to the free Euclidean field of mass \( m \) is the Gaussian process over \( S \) of mean 0 and covariance \( (g, (—\Delta + m^2)^{-1}f)_{L^2(\mathbb{R}^d)} \). According to a theorem of Minlos [4], this process may be realized on \( S' \), the topological dual of \( S \), by \( \varphi(f) : q \rightarrow \langle f, q \rangle \), where \( (\cdot, \cdot) \) denotes the \( S, S' \) pairing. That is, there is a Borel measure \( \mu \) on \( S' \) so that \( \varphi \) maps \( S \) to Gaussian random variables over \( S' \) with mean 0 and above specified covariances.

We will say that \( \mu \) is supported on a Hilbert space \( H \) if \( H \subset S' \), the injection of \( H \) into \( S' \) is continuous, and there is a Borel measure \( \mu_0 \) on \( H \) so that the restriction of \( \varphi(f) \) to \( H \) realizes on \( (H, \mu_0) \) the Gaussian process over \( S \) of mean 0 and specified covariance. Support properties of \( \mu \) have been studied by Reed and Rosen [8]. They utilized a theorem of Minlos to show that certain \( H \)'s support \( \mu \), and they showed by a rather lengthy direct computation that others failed to support. Related results concerning the support of \( \mu \) may be found in the recent work of Cannon [1] and of Colella and Lanford [2]. In this note we show that Minlos' sufficient condition for \( \mu \) to be supported on \( H \) is, in fact, necessary.

Proposition. Let \( (\cdot, \cdot)_1 \) and \( (\cdot, \cdot)_2 \) be continuous inner products on \( S \) such that \( \|f\|_2 \leq c\|f\|_1 \) for some constant \( c \) and for all \( f \) in \( S \). Let \( H_1 \) and \( H_2 \) be the Hilbert space completions of \( S \) with respect to \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \) respectively. Since \( S \) is separable, each \( H_i \) is also separable. Let \( (\varphi, \mu) \) be the realization on \( S' \) of the Gaussian process over \( S \) of mean 0 and covariance \( (g, f)_{L^2(\mathbb{R}^d)} \). Then \( H_1 \) supports \( \mu \) if and only if the natural injection \( H_1 \subset H_2 \) is Hilbert-Schmidt.
Remark 1. The following are equivalent definitions of the natural injection $\mathcal{H}_1 \subset \mathcal{H}_2$ being $\mathcal{H}$-S (Hilbert-Schmidt): (i) (used by Gelfand-Vilenkin [4]) There exists a positive symmetric $\mathcal{H}$-S operator $A_1$ on $\mathcal{H}_1$ such that $(f, g)_2 = (A_1 f, A_1 g)_1$. (ii) (used by Reed-Rosen [8]) There exists a 1-1 $\mathcal{H}$-S operator $A_2$ on $\mathcal{H}_2$ such that $\mathcal{S} \subset A_2 \mathcal{H}_2$ and $\mathcal{H}_1$ is the set $A_2 \mathcal{H}_2$ with norm $\|f\|_1 = \|A_2^{-1/2} f\|_2$. (iii) There exists a positive symmetric $\mathcal{H}$-S operator $A_3$ on $\mathcal{H}_2'$ such that $(\langle q, p \rangle_1, \langle q, p \rangle_2)_{1'} = (A_3 q, A_3 p)_{2'}$.

Remark 2. $\mathcal{S}$ may be replaced by a nuclear space $\mathcal{S}$.

Proof of Proposition. We make the identifications by injection $\mathcal{S} \subset \mathcal{H}_1 \subset \mathcal{H}_2$, by restriction $\mathcal{H}_2' \subset \mathcal{H}_1' \subset \mathcal{S}'$ and also canonically identify $\mathcal{H}_i$ with $\mathcal{H}_i''$ $(i = 1, 2)$. The map $\varphi(f)q \rightarrow \langle f, q \rangle$ furnishes a densely defined linear mapping of $\mathcal{H}_1''$ or $\mathcal{H}_2''$ to Gaussian random variables of mean 0 and covariance specified by the $\mathcal{H}_i''$ inner product. Since the $\mathcal{H}_2''$ inner product is continuous on $\mathcal{H}_1''$, we obtain a weak distribution [5] over $\mathcal{H}_1'$ which is the restriction to $\mathcal{H}_1'$ of the unit normal distribution over $\mathcal{H}_2'$. This weak distribution uniquely determines a (finitely additive) cylinder set measure $\mu_1$ on $\mathcal{H}_1'$.

$\mathcal{H}_1'$ supports $\mu$ if and only if $\mu_1$ is countably additive on the ring of cylinder sets in $\mathcal{H}_1'$. The separability of the Hilbert spaces allows us to apply a theorem of Dudley, Feldman and LeCam [3], which asserts that countable additivity of $\mu_1$ is equivalent to the pair $(\mathcal{H}_1', \mathcal{H}_1')$ forming an abstract Wiener space in the sense of L. Gross [5]. It is well known (and very easy to calculate) that if $\mathcal{K}_1$ and $\mathcal{K}_2$ are two real separable Hilbert spaces, then $(\mathcal{K}_1, \mathcal{K}_2)$ forms an abstract Wiener space if $\mathcal{K}_2$ is the completion of $\mathcal{K}_1$ with respect to an inner product $(f, g)_2 = (A_1 f, A_1 g)_1$, where $A_1$ is positive $\mathcal{H}$-S on $\mathcal{K}_1$.

Conversely, we claim that if a real separable Hilbert space $\mathcal{K}_2$ arises as the completion of a Hilbert space $\mathcal{K}_1$ with respect to a continuous norm $\|\cdot\|_2$ on $\mathcal{K}_1$, and if the pair $(\mathcal{K}_1, \mathcal{K}_2)$ forms an abstract Wiener space, then $(f, g)_2 = (A f, A g)_1$ where $A$ is positive, symmetric and $\mathcal{H}$-S on $\mathcal{K}_1$. Let us make the identifications $\mathcal{K}_2' \subset \mathcal{K}_1' \approx \mathcal{K}_1 \subset \mathcal{K}_2$, the first containment by restriction and the second by the canonical injection. Then it follows from [5, Corollary 5] that the canonical isomorphism $M$ of $\mathcal{K}_2$ onto $\mathcal{K}_2'$ has the property that when restricted to $\mathcal{K}_1$ and viewed as an operator $M_1$ on $\mathcal{K}_1$, it is positive symmetric and of trace class. But this means that $(f, g)_2 = (f, M_1 g)_1 = (\sqrt{M}_1 f, \sqrt{M}_1 g)_1$, where $\sqrt{M}_1$ is $\mathcal{H}$-S.

REFERENCES


