PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

SUPPORT PROPERTIES OF GAUSSIAN PROCESSES OVER SCHWARTZ SPACE[1]

M. ANN PIECH

ABSTRACT. We utilize the concept of an abstract Wiener space to prove a converse to a theorem of Minlos, thereby obtaining necessary and sufficient conditions for a Hilbert subspace of $\mathcal{S}'(\mathbf{R}^d)$ to support a given Gaussian process over $\mathcal{S}(\mathbf{R}^d)$.

Stochastic processes over $\mathcal{S}(\mathbf{R}^d)$ are of current interest as elements in the construction of relativistic Boson field theories (see Nelson [6], [7]). The basic process corresponding to the free Euclidean field of mass m is the Gaussian process over $\mathcal{S}$ of mean 0 and covariance $(g, (-\Delta + m^2)^{-1}f)_{L^2(\mathbf{R}^d)}$. According to a theorem of Minlos [4], this process may be realized on $\mathcal{S}'$, the topological dual of $\mathcal{S}$, by $\varphi(f)$: $q \to \langle f, q\rangle$, where $\langle\ ,\ \rangle$ denotes the $\mathcal{S}$-$\mathcal{S}'$ pairing. That is, there is a Borel measure μ on $\mathcal{S}'$ so that φ maps $\mathcal{S}$ to Gaussian random variables over $(\mathcal{S}', \mu)$ with mean 0 and above specified covariances.

We will say that μ is supported on a Hilbert space $\mathcal{H}$ if $\mathcal{H} \subset \mathcal{S}'$, the injection of $\mathcal{H}$ into $\mathcal{S}'$ is continuous, and there is a Borel measure μ_0 on $\mathcal{H}$ so that the restriction of $\varphi(f)$ to $\mathcal{H}$ realizes on $(\mathcal{H}, \mu_0)$ the Gaussian process over $\mathcal{S}$ of mean 0 and specified covariance. Support properties of μ have been studied by Reed and Rosen [8]. They utilized a theorem of Minlos to show that certain $\mathcal{H}$'s support μ, and they showed by a rather lengthy direct computation that others failed to support. Related results concerning the support of μ may be found in the recent work of Cannon [1] and of Colella and Lanford [2]. In this note we show that Minlos' sufficient condition for μ to be supported on $\mathcal{H}$ is, in fact, necessary.

Proposition. *Let $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$ be continuous inner products on $\mathcal{S}$ such that $\|f\|_2 \le c\|f\|_1$ for some constant c and for all f in $\mathcal{S}$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert space completions of $\mathcal{S}$ with respect to $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. Since $\mathcal{S}$ is separable, each $\mathcal{H}_i$ is also separable. Let (φ, μ) be the realization on $\mathcal{S}'$ of the Gaussian process over $\mathcal{S}$ of mean 0 and covariance $(g, f)_2$. Then $\mathcal{H}_1'$ supports μ if and only if the natural injection $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ is Hilbert-Schmidt.*

Received by the editors August 7, 1974 and, in revised form, October 9, 1974.
AMS (MOS) subject classifications (1970). Primary 60G15; Secondary 81A18; 28A40.
Key words and phrases. Gaussian process, Schwartz space, free Boson field, abstract Wiener space.
[1]Research supported by NSF grant PO28934.

Remark 1. The following are equivalent definitions of the natural injection $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ being $\mathcal{H}$-$\mathcal{S}$ (Hilbert-Schmidt): (i) (used by Gelfand-Vilenkin [4]) There exists a positive symmetric $\mathcal{H}$-$\mathcal{S}$ operator A_1 on $\mathcal{H}_1$ such that $(f, g)_2 = (Af, Ag)_1$. (ii) (used by Reed-Rosen [8]) There exists a 1-1 $\mathcal{H}$-$\mathcal{S}$ operator A_2 on $\mathcal{H}_2$ such that $\mathcal{S} \subset A_2\mathcal{H}_2$ and $\mathcal{H}_1$ is the set $A_2\mathcal{H}_2$ with norm $\|f\|_1 = \|A_2^{-1}f\|_2$. (iii) There exists a positive symmetric $\mathcal{H}$-$\mathcal{S}$ operator A_3 on $\mathcal{H}'_2$ such that $(q, p)_{1'} = (A_3q, A_3p)_{2'}$.

Remark 2. $\mathcal{S}$ may be replaced by a nuclear space $\mathfrak{S}$.

Proof of Proposition. We make the identifications by injection $\mathcal{S} \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_2$, by restriction $\mathcal{H}'_2 \subseteq \mathcal{H}'_1 \subseteq \mathcal{S}'$ and also canonically identify $\mathcal{H}_i$ with $\mathcal{H}''_i$ $(i = 1, 2)$. The map $\varphi(f)q \longrightarrow \langle f, q\rangle$ furnishes a densely defined linear mapping of $\mathcal{H}''_1$ or $\mathcal{H}''_2$ to Gaussian random variables of mean 0 and covariance specified by the $\mathcal{H}''_2$ inner product. Since the $\mathcal{H}''_2$ inner product is continuous on $\mathcal{H}''_1$, we obtain a weak distribution [5] over $\mathcal{H}'_1$ which is the restriction to $\mathcal{H}''_1$ of the unit normal distribution over $\mathcal{H}'_2$. This weak distribution uniquely determines a (finitely additive) cylinder set measure μ_1 on $\mathcal{H}'_1$.

$\mathcal{H}'_1$ supports μ if and only if μ_1 is countably additive on the ring of cylinder sets in $\mathcal{H}'_1$. The separability of the Hilbert spaces allows us to apply a theorem of Dudley, Feldman and LeCam [3], which asserts that countable additivity of μ_1 is equivalent to the pair $(\mathcal{H}'_2, \mathcal{H}'_1)$ forming an abstract Wiener space in the sense of L. Gross [5]. It is well known (and very easy to calculate) that if $\mathcal{K}_1$ and $\mathcal{K}_2$ are two real separable Hilbert spaces, then $(\mathcal{K}_1, \mathcal{K}_2)$ forms an abstract Wiener space if $\mathcal{K}_2$ is the completion of $\mathcal{K}_1$ with respect to an inner product $(f, g)_2 = (Af, Ag)_1$, where A is positive $\mathcal{H}$-$\mathcal{S}$ on $\mathcal{K}_1$.

Conversely, we claim that if a real separable Hilbert space $\mathcal{K}_2$ arises as the completion of a Hilbert space $\mathcal{K}_1$ with respect to a continuous norm $\|\cdot\|_2$ on $\mathcal{K}_1$, and if the pair $(\mathcal{K}_1, \mathcal{K}_2)$ forms an abstract Wiener space, then $(f, g)_2 = (Af, Ag)_1$ where A is positive, symmetric and $\mathcal{H}$-$\mathcal{S}$ on $\mathcal{K}_1$. Let us make the identifications $\mathcal{K}'_2 \subseteq \mathcal{K}'_1 \approx \mathcal{K}_1 \subseteq \mathcal{K}_2$, the first containment by restriction and the second by the canonical injection. Then it follows from [5, Corollary 5] that the canonical isomorphism M of $\mathcal{K}_2$ onto $\mathcal{K}'_2$ has the property that when restricted to $\mathcal{K}_1$ and viewed as an operator M_1 on $\mathcal{K}_1$, it is positive symmetric and of trace class. But this means that $(f, g)_2 = (f, M_1g)_1 = (\sqrt{M}_1f, \sqrt{M}_1g)_1$, where $\sqrt{M}_1$ is $\mathcal{H}$-$\mathcal{S}$.

REFERENCES

1. J. Cannon, *Continuous sample paths in quantum field theory*, Comm. Math. Phys. 35 (1974), 215–233.

2. P. Colella and O. Lanford, *Sample field behavior for the free Markov random field*, Lecture Notes in Physics, vol. 25, Constructive Quantum Field Theory, Springer-Verlag, Berlin and New York, 1973.

3. R. M. Dudley, J. Feldman and L. Le Cam, *On seminorms and probabilities, and abstract Wiener spaces*, Ann. of Math. (2) 93 (1971), 390–408. MR 43 #4995.

4. I. M. Gel′fand and N. Ja. Vilenkin, *Generalized functions*. Vol. 4: *Some applications of harmonic analysis*, Fizmatgiz, Moscow, 1961; English transl., Academic Press, New York, 1964. MR 26 #4173; 30 #4152.

5. L. Gross, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. II: Contributions to Probability Theory, part I, Univ. California Press, Berkeley, Calif., 1967, pp. 31–42. MR 35 #3027.

6. E. Nelson, *Construction of quantum fields from Markoff fields*, J. Functional Analysis 12 (1973), 97–112.

7. ———, *Probability theory and Euclidean field theory*, Lecture Notes in Physics, vol. 25, Constructive Quantum Field Theory, Springer-Verlag, Berlin and New York, 1973.

8. M. Reed and L. Rosen, *Support properties of the free measure for Boson fields*, Comm. Math. Phys. 36 (1974), 123–132.

DEPARTMENT OF MATHEMATICS, SUNY AT BUFFALO, AMHERST, NEW YORK 14226