

## RANDOM COMPACT SETS RELATED TO THE KAKEYA PROBLEM

RALPH ALEXANDER

ABSTRACT. A  $B$ -set is defined to be a compact planar set of zero measure which contains a translate of any line segment lying in a disk of diameter one. A construction is given which associates a unique compact planar set with each sequence in a closed interval, and it is shown that for almost all such sequences a  $B$ -set is obtained. The construction depends on the measure properties of certain perfect linear sets. Several related problems of a subtler nature are also considered.

1. **Introduction.** Long ago Besicovitch [1] gave his famous example of a compact planar set of measure zero which contains a translate of every line segment lying in a disk of diameter one. For convenience we will call such a set a  $B$ -set. Although the original construction of Besicovitch was rather complicated, there have been a number of elegant simplifications, especially for the construction of sets of measure  $\epsilon$  containing the required line segments. The idea of Schoenberg discussed in [3] is particularly successful.

In this article we give a simple probabilistic method for generating a large family of  $B$ -sets. We only need elementary results about the measure of certain linear sets, and a rudimentary knowledge of random sequences.

There are a number of subtle questions which do arise, however. We are able to deal with several of these by appealing to a deep theorem of Besicovitch [2] concerning planar sets of finite Carathéodory length.

2. **The measure of certain linear sets.** Let  $a$ ,  $b$  and  $x_1$  lie in the interval  $[0, 2/3]$ . Consider the three closed intervals  $[a, a + 1/3]$ ,  $[b, b + 1/3]$ , and  $[x_1, x_1 + 1/3]$ . Let  $K(x_1)$  denote this collection of intervals, and let  $T(x_1)$  denote their union.

We will form the collection  $K(x_1, x_2)$ , consisting of nine intervals of length  $1/9$ , as follows: For each member  $[y, y + 1/3]$  in  $K(x_1)$ , there is precisely one affine transformation  $\tau$  of the line such that  $\tau(0) = y$  and  $\tau(1) = y + 1/3$ . The images under  $\tau$  of the three intervals in  $K(x_2)$  will be three intervals of length  $1/9$  lying in  $[y, y + 1/3]$ . Applying this construction to each of the three intervals in  $K(x_1)$  yields the nine intervals in  $K(x_1, x_2)$ .

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Inductively, if  $a, b, x_1, \dots, x_n$  are numbers in  $[0, 2/3]$ ,  $K(x_1, \dots, x_n)$  will consist of  $3^n$  intervals of length  $3^{-n}$ . For each interval  $[y, y + 3^{-n+1}]$  in  $K(x_1, \dots, x_{n-1})$ , there will be an affine transformation  $\tau$  such that  $\tau(0) = y$  and  $\tau(1) = y + 3^{-n+1}$ . The images of the three intervals in  $K(x_n)$  will be three intervals in  $[y, y + 3^{-n+1}]$ . Thus we obtain the  $3^n$  intervals in  $K(x_1, \dots, x_n)$ .  $T(x_1, \dots, x_n)$  will denote the union of these intervals, and  $l(x_1, \dots, x_n)$  will denote the linear measure of this union. Clearly,  $T(x_1, \dots, x_n) \subset T(x_1, \dots, x_{n-1})$  so that  $l(x_1, \dots, x_n) \leq l(x_1, \dots, x_{n-1})$ .

If  $a, b, x_1, x_2, \dots$  is a sequence in  $[0, 2/3]$ , we define  $T(x_1, x_2, \dots)$  to be  $\bigcap_n T(x_1, \dots, x_n)$  and  $l(x_1, x_2, \dots)$  to be the measure of  $T(x_1, x_2, \dots)$ . We note that when  $a = 0$  and  $b = 2/3$ ,  $T(1/3, 1/3, \dots)$  is the entire unit interval, while  $T(0, 0, \dots)$  is the usual Cantor set. We also observe that if  $x_1, x_2 = \dots = x_n = a$ , then  $l(x_1, \dots, x_n) \leq (2/3)^n$ , since  $T(x_1, \dots, x_n)$  can be expressed as the union of  $2^n$  intervals of length  $3^{-n}$ .

**Lemma 1.** *Let  $a, b, x_1, x_2, \dots$  be a sequence in  $[0, 2/3]$ . Then for any positive integers  $m, n$  we have the inequalities*

$$(1) \quad l(x_1, x_2, \dots) < l(x_{m+1}, x_{m+2}, \dots) \leq l(x_{m+1}, \dots, x_{m+n}).$$

**Proof.** It is clear from the definitions that  $l(x_{m+1}, x_{m+2}, \dots) \leq l(x_{m+1}, \dots, x_{m+n})$ . We note that  $T(x_1, x_2, \dots)$  is the union of  $3^m$  similar images of  $T(x_{m+1}, x_{m+2}, \dots)$  and the ratio of similarity is  $3^{-m}$  for each image. The inequality  $l(x_1, x_2, \dots) \leq l(x_{m+1}, x_{m+2}, \dots)$  follows at once.

**Lemma 2.** *Let  $a$  and  $b$  be numbers in the interval  $[0, 2/3]$ . The function  $l(x_1, \dots, x_n)$  from  $[0, 2/3]^n$  to the interval  $[0, 1]$  is continuous.*

The lemma is obviously true and we omit a proof. Questions concerning the modulus of continuity of  $l$  seem difficult, however.

**Proposition 1.** *Almost all sequences  $x_1, x_2, \dots$  in  $[0, 2/3]$  have the property that given any  $a$  and  $b$  in  $[0, 2/3]$ , then  $l(x_1, x_2, \dots) = 0$ .*

**Proof.** It follows from the classical results of E. Borel and others that given any positive number  $\delta$  and positive integer  $n$ , almost all sequences in  $[0, 2/3]$  have the property that for any number  $a$  in  $[0, 2/3]$ , the inequality  $|x_i - a| < \delta$  will be satisfied for at least  $n$  successive values of  $i$ . (See [4, Problem 5, p. 197].)

Now by Lemma 2 there is a  $\delta$  such that if  $|x_i - a| < \delta$  for  $i = m + 1, \dots, m + n$ , then  $l(x_{m+1}, \dots, x_{m+n}) < (2/3)^n + \epsilon$ . However, it follows from inequality (1) that  $l(x_1, x_2, \dots) < (2/3)^n + \epsilon$ . Since  $n$  can be arbitrarily large,  $l(x_1, x_2, \dots) = 0$ . This concludes the proof.

We remark that "good" sequences are easy to find. For example, we could choose  $x_i$  to be the fractional part of  $(2/3)(\sum_{j=1}^i 1/j)$ . The method of

proof for Proposition 1 makes it clear that for any  $a$  and  $b$  in  $[0, 2/3]$ ,  $l(x_1, x_2, \dots) = 0$ .

The value of  $l(x_1, x_2, \dots)$  for a specified sequence is generally impossible to determine. Using a devious argument, which will be outlined later, we are able to state the following nonelementary result.

**Proposition 2.** *Let  $a$  and  $b$  be numbers in the interval  $[0, 2/3]$ . For almost all  $x$  in  $[0, 2/3]$ ,  $l(x, x, \dots) = 0$ .*

**3. A planar construction.** We now do some analogous constructions in the plane. Let  $x_1$  be a number in  $[0, 2/3]$ . By  $K^*(x_1)$  we denote the set of three closed parallelograms whose vertices in clockwise order are  $(0, 0)$ ,  $(2/3, 1)$ ,  $(1, 1)$ ,  $(1/3, 0)$ ;  $(2/3, 0)$ ,  $(0, 1)$ ,  $(1/3, 1)$ ,  $(1, 0)$ ;  $(x_1, 0)$ ,  $(x_1, 1)$ ,  $(x_1 + 1/3, 1)$ ,  $(x_1 + 1/3, 0)$ .  $T^*(x_1)$  will denote the union of these three parallelograms.

If  $I$  denotes the unit square  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 1)$ ,  $(1, 0)$  and  $P$  is one of the parallelograms in  $K^*(x_1)$ , then there is a unique affine transformation  $\tau$  of the plane which sends the vertices of  $I$  to the corresponding vertices of  $P$  in the given order. The set  $K^*(x_1, x_2)$  will consist of nine parallelograms of area  $1/9$  which are the images of the members of  $K^*(x_2)$  under the three  $\tau$ 's associated with  $K^*(x_1)$ .

In general,  $K^*(x_1, \dots, x_n)$  will consist of  $3^n$  parallelograms of area  $3^{-n}$ . For each of the  $3^{n-1}$  parallelograms  $P$  in  $K^*(x_1, \dots, x_{n-1})$  there is an affine  $\tau$  taking  $I$  to  $P$  with proper vertices corresponding. The set  $K^*(x_1, \dots, x_n)$  consists of the various images of the members of  $K^*(x_n)$  under these transformations. We let  $T^*(x_1, \dots, x_n)$  denote the union of the members of  $K^*(x_1, \dots, x_n)$  and let  $l^*(x_1, \dots, x_n)$  denote the planar measure of this union.

If  $x_1, x_2, \dots$  is a sequence in  $[0, 2/3]$ , we define  $T^*(x_1, x_2, \dots)$  and  $l^*(x_1, x_2, \dots)$  analogously to their linear counterparts. Also, let  $T'(x_1, x_2, \dots)$  be the planar set obtained by rotating  $T^*(x_1, x_2, \dots)$  through a positive angle of  $\pi/2$  about  $(1/2, 1/2)$ , the center of  $I$ .

**Lemma 3.** *If  $x_1, x_2, \dots$  is any sequence in  $[0, 2/3]$ , the planar set  $T^*(x_1, x_2, \dots) \cup T'(x_1, x_2, \dots)$  contains a translate of any line segment lying in the unit square  $I$ .*

**Proof.** Let  $L$  be a line segment in  $I$ . Let us suppose that the line determined by  $L$  and the  $x$ -axis determine an angle (measured from axis to line) in the interval  $[\pi/4, 3\pi/4]$ . We may assume that  $L$  joins a point on the top edge of  $I$  to a point on the bottom edge. It is easy to see that at least one  $P$  in  $K^*(x_1)$  contains a translate of  $L$ . Because affine transformations preserve parallelism, it is seen that  $T^*(x_1, \dots, x_n)$  will also contain a translate of  $L$ . By standard arguments, the compact set  $T^*(x_1, x_2, \dots)$  will contain a translate of  $L$ .

If  $L$  and the  $x$ -axis determine an angle in  $[-\pi/4, \pi/4]$ , then  $T'(x_1, x_2, \dots)$  will contain a translate of  $L$ . This completes the proof.

**Proposition 3.** *For almost all sequences  $x_1, x_2, \dots$  in  $[0, 2/3]$  the planar set  $T^*(x_1, x_2, \dots) \cup T'(x_1, x_2, \dots)$  is a B-set.*

**Proof.** For  $0 \leq t \leq 1$  let  $L_t$  be the horizontal line segment joining the points  $(0, t)$  and  $(1, t)$ . We observe that the  $y$ -section of  $T^*(x_1, x_2, \dots)$  determined by  $L_t$  is of the form  $T(x_1, x_2, \dots)$  where  $a = (2/3)t$  and  $b = (2/3)(1 - t)$ . The sequence  $x_1, x_2, \dots$  remains unchanged. Thus for almost all sequences  $x_1, x_2, \dots$  every  $y$ -section of  $T^*(x_1, x_2, \dots)$  has linear measure zero. The result follows at once.

We now state a much deeper result corresponding to Proposition 2.

**Proposition 4.** *If  $x$  is any number in  $[0, 2/3]$ , except  $1/3$ , then  $l^*(x, x, \dots) = 0$ ,  $l^*(1/3, 1/3, \dots) = 1/2$ .*

**4. Outline of proofs for Propositions 2 and 4.** Our next lemma shows that in studying the behaviour of  $l(x, x, \dots)$  we need only consider the case  $a = 0$  and  $b = 2/3$ .

**Lemma 4.** *Let  $a, b$  and  $x$  be numbers in the interval  $[0, 2/3]$ . Then  $T(x, x, \dots)$  is similar to  $T(x', x', \dots)$  where  $a = 0$  and  $b = 2/3$ .*

**Proof.** We note that the set  $T(x, x, \dots)$  is the union of three similar images of itself, the ratio of similarity being  $1/3$ . Call these images  $T_1, T_2$  and  $T_3$ , and let  $y_1, y_2$  and  $y_3$  be their respective least members. We may assume that  $y_1 \leq y_2 \leq y_3$ . If  $z$  is the largest member of  $T(x, x, \dots)$ , then there is a unique affine transformation  $\tau$  such that  $\tau(y_1) = 0$  and  $\tau(z) = 1$ . It follows that  $\tau(y_3) = 2/3$ ; we define  $x' = \tau(y_2)$ . It is apparent that  $\tau(T(x, x, \dots)) = T(x', x', \dots)$ .

If  $a \leq x \leq b$ , we can see that  $y_1 = 3a/2, y_2 = x + y_1/3, y_3 = b + y_1/3, z = 1 - 3(1 - b)/2$ . It is interesting to observe that if  $i \neq j, T_i \cap T_j$  has linear measure zero even though  $T(x, x, \dots)$  may not. From this point on we will always assume that  $a = 0$  and  $b = 2/3$ .

Our proofs of Propositions 2 and 4 depend on the projections of a planar set  $E$  which we now define. Let  $E_0$  be an equilateral triangle of side one. The collection  $E_1$  will consist of the three homothets of  $E_0$  of side  $1/3$  obtained by dilations of ratio  $1/3$  centered at each of the three vertices of  $E_0$ . We obtain  $E_2$ , a collection of nine equilateral triangles of side  $1/9$ , by performing dilations of ratio  $1/3$  centered at each vertex of each member of  $E_1$  so that a member of  $E_1$  gives rise to three triangles in  $E_2$ . We proceed inductively, and, in general,  $E_n$  will consist of  $3^n$  equilateral triangles of side  $3^{-n}$ . Let  $E = \bigcap E_n$ .

For those familiar with Besicovitch's theory of planar sets of finite

length, it is easy to see that  $E$  is an irregular set of Carathéodory length 1. The fundamental theorem of Besicovitch [2] assures that the almost all directions  $\theta$ , the linear orthogonal projection  $E_\theta$  of  $E$  possesses zero linear measure.

For each  $\theta$ ,  $E_\theta$  is similar to  $T(x, x, \dots)$  for a suitable  $x$ . In fact, we need only consider certain  $\theta$ -intervals of length  $\pi/6$  to be assured that for each  $x$  a similar image  $E_{\theta(x)}$  occurs. Furthermore, it is clear that  $x$  and  $\theta(x)$  are related in an absolutely bicontinuous manner over any  $\theta$ -interval in which the mapping  $x \rightarrow \theta(x)$  is one to one. It follows that  $l(x, x, \dots) = 0$  for almost all  $x$  in  $[0, 2/3]$ .

Proposition 4 is established in a similar manner by relating the  $y$ -sections of  $T^*(x, x, \dots)$ ,  $0 \leq y \leq 1$ , to  $\theta^*(y)$  where  $E_{\theta^*(y)}$  is similar to the  $y$ -section of  $T^*(x, x, \dots)$ . This can easily be done in every case, except  $x = 1/3$ , to show that  $l^*(x, x, \dots) = 0$ . When  $x = 1/3$ , the mapping  $y \rightarrow \theta^*(y)$  is constant. In fact each  $y$ -section is similar to  $[0, 1]$ . The set  $T^*(1/3, 1/3, \dots)$  consists of the two triangles with vertices  $(0, 1)$ ,  $(1, 1)$ ,  $(1/2, 1/2)$  and  $(0, 0)$ ,  $(1, 0)$ ,  $(1/2, 1/2)$ .

We admit that the method of proof outlined above is somewhat artificial and does not readily generalize to constructions involving more than three intervals. We hope that a direct proof of Proposition 2 can be found which will tell precisely for which  $x$  it is true that  $l(x, x, \dots) = 0$ .

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS, 61801