PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

RANDOM COMPACT SETS RELATED TO THE KAKEYA PROBLEM

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ABSTRACT. A B-set is defined to be a compact planar set of zero measure which contains a translate of any line segment lying in a disk of diameter one. A construction is given which associates a unique compact planar set with each sequence in a closed interval, and it is shown that for almost all such sequences a B-set is obtained. The construction depends on the measure properties of certain perfect linear sets. Several related problems of a subtler nature are also considered.

1. **Introduction.** Long ago Besicovitch [1] gave his famous example of a compact planar set of measure zero which contains a translate of every line segment lying in a disk of diameter one. For convenience we will call such a set a B-set. Although the original construction of Besicovitch was rather complicated, there have been a number of elegant simplifications, especially for the construction of sets of measure ϵ containing the required line segments. The idea of Schoenberg discussed in [3] is particularly successful.

In this article we give a simple probabilistic method for generating a large family of B-sets. We only need elementary results about the measure of certain linear sets, and a rudimentary knowledge of random sequences.

There are a number of subtle questions which do arise, however. We are able to deal with several of these by appealing to a deep theorem of Besicovitch [2] concerning planar sets of finite Carathéodory length.

2. **The measure of certain linear sets.** Let a, b and x_1 lie in the interval $[0, 2/3]$. Consider the three closed intervals $[a, a+1/3]$, $[b, b+1/3]$, and $[x_1, x_1+1/3]$. Let $K(x_1)$ denote this collection of intervals, and let $T(x_1)$ denote their union.

We will form the collection $K(x_1, x_2)$, consisting of nine intervals of length $1/9$, as follows: For each member $[y, y+1/3]$ in $K(x_1)$, there is precisely one affine transformation τ of the line such that $\tau(0)=y$ and $\tau(1)=y+1/3$. The images under τ of the three intervals in $K(x_2)$ will be three intervals of length $1/9$ lying in $[y, y+1/3]$. Applying this construction to each of the three intervals in $K(x_1)$ yields the nine intervals in $K(x_1, x_2)$.

Received by the editors October 14, 1974.
AMS (MOS) subject classifications (1970). Primary 28A75.

Inductively, if a, b, $x_1, \ldots, x_n$ are numbers in $[0, 2/3]$, $K(x_1, \ldots, x_n)$ will consist of 3^n intervals of length 3^{-n}. For each interval $[y, y+3^{-n+1}]$ in $K(x_1, \ldots, x_{n-1})$, there will be an affine transformation τ such that $\tau(0) = y$ and $\tau(1) = y + 3^{-n+1}$. The images of the three intervals in $K(x_n)$ will be three intervals in $[y, y+3^{-n+1}]$. Thus we obtain the 3^n intervals in $K(x_1, \ldots, x_n)$. $T(x_1, \ldots, x_n)$ will denote the union of these intervals, and $l(x_1, \ldots, x_n)$ will denote the linear measure of this union. Clearly, $T(x_1, \ldots, x_n) \subset T(x_1, \ldots, x_{n-1})$ so that $l(x_1, \ldots, x_n) \leq l(x_1, \ldots, x_{n-1})$.

If a, b, $x_1, x_2, \ldots$ is a sequence in $[0, 2/3]$, we define $T(x_1, x_2, \ldots)$ to be $\bigcap_n T(x_1, \ldots, x_n)$ and $l(x_1, x_2, \ldots)$ to be the measure of $T(x_1, x_2, \ldots)$. We note that when $a = 0$ and $b = 2/3$, $T(1/3, 1/3, \ldots)$ is the entire unit interval, while $T(0, 0, \ldots)$ is the usual Cantor set. We also observe that if $x_1, x_2 = \cdots = x_n = a$, then $l(x_1, \ldots, x_n) \leq (2/3)^n$, since $T(x_1, \ldots, x_n)$ can be expressed as the union of 2^n intervals of length 3^{-n}.

Lemma 1. *Let a, b, $x_1, x_2, \ldots$ be a sequence in $[0, 2/3]$. Then for any positive integers m, n we have the inequalities*

$$l(x_1, x_2, \ldots) < l(x_{m+1}, x_{m+2}, \ldots) \leq l(x_{m+1}, \ldots, x_{m+n}). \tag{1}$$

Proof. It is clear from the definitions that $l(x_{m+1}, x_{m+2}, \ldots) \leq l(x_{m+1}, \ldots, x_{m+n})$. We note that $T(x_1, x_2, \ldots)$ is the union of 3^m similar images of $T(x_{m+1}, x_{m+2}, \ldots)$ and the ratio of similarity is 3^{-m} for each image. The inequality $l(x_1, x_2, \ldots) \leq l(x_{m+1}, x_{m+2}, \ldots)$ follows at once.

Lemma 2. *Let a and b be numbers in the interval $[0, 2/3]$. The function $l(x_1, \ldots, x_n)$ from $[0, 2/3]^n$ to the interval $[0, 1]$ is continuous.*

The lemma is obviously true and we omit a proof. Questions concerning the modulus of continuity of l seem difficult, however.

Proposition 1. *Almost all sequences $x_1, x_2, \ldots$ in $[0, 2/3]$ have the property that given any a and b in $[0, 2/3]$, then $l(x_1, x_2, \ldots) = 0$.*

Proof. It follows from the classical results of E. Borel and others that given any positive number δ and positive integer n, almost all sequences in $[0, 2/3]$ have the property that for any number a in $[0, 2/3]$, the inequality $|x_i - a| < \delta$ will be satisfied for at least n successive values of i. (See [4, Problem 5, p. 197].)

Now by Lemma 2 there is a δ such that if $|x_i - a| < \delta$ for $i = m+1, \ldots, m+n$, then $l(x_{m+1}, \ldots, x_{m+n}) < (2/3)^n + \epsilon$. However, it follows from inequality (1) that $l(x_1, x_2, \ldots) < (2/3)^n + \epsilon$. Since n can be arbitrarily large, $l(x_1, x_2, \ldots) = 0$. This concludes the proof.

We remark that "good" sequences are easy to find. For example, we could choose x_i to be the fractional part of $(2/3)(\sum_{j=1}^{i} 1/j)$. The method of proof for Proposition 1 makes it clear that for any a and b in $[0, 2/3]$, $l(x_1, x_2, \ldots) = 0$.

The value of $l(x_1, x_2, \ldots)$ for a specified sequence is generally impossible to determine. Using a devious argument, which will be outlined later, we are able to state the following nonelementary result.

Proposition 2. *Let a and b be numbers in the interval $[0, 2/3]$. For almost all x in $[0, 2/3]$, $l(x, x, \ldots) = 0$.*

3. A planar construction. We now do some analogous constructions in the plane. Let x_1 be a number in $[0, 2/3]$. By $K^*(x_1)$ we denote the set of three closed parallelograms whose vertices in clockwise order are $(0, 0)$, $(2/3, 1)$, $(1, 1)$, $(1/3, 0)$; $(2/3, 0)$, $(0, 1)$, $(1/3, 1)$, $(1, 0)$; $(x_1, 0)$, $(x_1, 1)$, $(x_1 + 1/3, 1)$, $(x_1 + 1/3, 0)$. $T^*(x_1)$ will denote the union of these three parallelograms.

If I denotes the unit square $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 0)$ and P is one of the parallelograms in $K^*(x_1)$, then there is a unique affine transformation τ of the plane which sends the vertices of I to the corresponding vertices of P in the given order. The set $K^*(x_1, x_2)$ will consist of nine parallelograms of area $1/9$ which are the images of the members of $K^*(x_2)$ under the three τ's associated with $K^*(x_1)$.

In general, $K^*(x_1, \ldots, x_n)$ will consist of 3^n parallelograms of area 3^{-n}. For each of the 3^{n-1} parallelograms P in $K^*(x_1, \ldots, x_{n-1})$ there is an affine τ taking I to P with proper vertices corresponding. The set $K^*(x_1, \ldots, x_n)$ consists of the various images of the members of $K^*(x_n)$ under these transformations. We let $T^*(x_1, \ldots, x_n)$ denote the union of the members of $K^*(x_1, \ldots, x_n)$ and let $l^*(x_1, \ldots, x_n)$ denote the planar measure of this union.

If $x_1, x_2, \ldots$ is a sequence in $[0, 2/3]$, we define $T^*(x_1, x_2, \ldots)$ and $l^*(x_1, x_2, \ldots)$ analogously to their linear counterparts. Also, let $T'(x_1, x_2, \ldots)$ be the planar set obtained by rotating $T^*(x_1, x_2, \ldots)$ through a positive angle of $\pi/2$ about $(1/2, 1/2)$, the center of I.

Lemma 3. *If $x_1, x_2, \ldots$ is any sequence in $[0, 2/3]$, the planar set $T^*(x_1, x_2, \ldots) \cup T'(x_1, x_2, \ldots)$ contains a translate of any line segment lying in the unit square I.*

Proof. Let L be a line segment in I. Let us suppose that the line determined by L and the x-axis determine an angle (measured from axis to line) in the interval $[\pi/4, 3\pi/4]$. We may assume that L joins a point on the top edge of I to a point on the bottom edge. It is easy to see that at least one P in $K^*(x_1)$ contains a translate of L. Because affine transformations preserve parallelism, it is seen that $T^*(x_1, \ldots, x_n)$ will also contain a translate of L. By standard arguments, the compact set $T^*(x_1, x_2, \ldots)$ will contain a translate of L.

If L and the x-axis determine an angle in $[-\pi/4, \pi/4]$, then $T'(x_1, x_2, \ldots)$ will contain a translate of L. This completes the proof.

Proposition 3. *For almost all sequences* $x_1, x_2, \ldots$ *in* $[0, 2/3]$ *the planar set* $T^*(x_1, x_2, \ldots) \cup T'(x_1, x_2, \ldots)$ *is a B-set.*

Proof. For $0 \le t \le 1$ let L_t be the horizontal line segment joining the points $(0, t)$ and $(1, t)$. We observe that the y-section of $T^*(x_1, x_2, \ldots)$ determined by L_t is of the form $T(x_1, x_2, \ldots)$ where $a = (2/3)t$ and $b = (2/3)(1-t)$. The sequence $x_1, x_2, \ldots$ remains unchanged. Thus for almost all sequences $x_1, x_2, \ldots$ every y-section of $T^*(x_1, x_2, \ldots)$ has linear measure zero. The result follows at once.

We now state a much deeper result corresponding to Proposition 2.

Proposition 4. *If* x *is any number in* $[0, 2/3]$, *except* $1/3$, *then* $l^*(x, x, \ldots) = 0$, $l^*(1/3, 1/3, \ldots) = 1/2$.

4. **Outline of proofs for Propositions 2 and 4.** Our next lemma shows that in studying the behaviour of $l(x, x, \ldots)$ we need only consider the case $a = 0$ and $b = 2/3$.

Lemma 4. *Let* a, b *and* x *be numbers in the interval* $[0, 2/3]$. *Then* $T(x, x, \ldots)$ *is similar to* $T(x', x', \ldots)$ *where* $a = 0$ *and* $b = 2/3$.

Proof. We note that the set $T(x, x, \ldots)$ is the union of three similar images of itself, the ratio of similarity being $1/3$. Call these images T_1, T_2 and T_3, and let y_1, y_2 and y_3 be their respective least members. We may assume that $y_1 \le y_2 \le y_3$. If z is the largest member of $T(x, x, \ldots)$, then there is a unique affine transformation τ such that $\tau(y_1) = 0$ and $\tau(z) = 1$. It follows that $\tau(y_3) = 2/3$; we define $x' = \tau(y_2)$. It is apparent that $\tau(T(x, x, \ldots)) = T(x', x', \ldots)$.

If $a \le x \le b$, we can see that $y_1 = 3a/2$, $y_2 = x + y_1/3$, $y_3 = b + y_1/3$, $z = 1 - 3(1-b)/2$. It is interesting to observe that if $i \ne j$, $T_i \cap T_j$ has linear measure zero even though $T(x, x, \ldots)$ may not. From this point on we will always assume that $a = 0$ and $b = 2/3$.

Our proofs of Propositions 2 and 4 depend on the projections of a planar set E which we now define. Let E_0 be an equilateral triangle of side one. The collection E_1 will consist of the three homothets of E_0 of side $1/3$ obtained by dilations of ratio $1/3$ centered at each of the three vertices of E_0. We obtain E_2, a collection of nine equilateral triangles of side $1/9$, by performing dilations of ratio $1/3$ centered at each vertex of each member of E_1 so that a member of E_1 gives rise to three triangles in E_2. We proceed inductively, and, in general, E_n will consist of 3^n equilateral triangles of side 3^{-n}. Let $E = \bigcap E_n$.

For those familiar with Besicovitch's theory of planar sets of finite length, it is easy to see that E is an irregular set of Carathéodory length 1. The fundamental theorem of Besicovitch [2] assures that the almost all directions θ, the linear orthogonal projection E_θ of E possesses zero linear measure.

For each θ, E_θ is similar to $T(x, x, \ldots)$ for a suitable x. In fact, we need only consider certain θ-intervals of length $\pi/6$ to be assured that for each x a similar image $E_{\theta(x)}$ occurs. Furthermore, it is clear that x and $\theta(x)$ are related in an absolutely bicontinuous manner over any θ-interval in which the mapping $x \longrightarrow \theta(x)$ is one to one. It follows that $l(x, x, \ldots) = 0$ for almost all x in $[0, 2/3]$.

Proposition 4 is established in a similar manner by relating the y-sections of $T^*(x, x, \ldots)$, $0 \leq y \leq 1$, to $\theta^*(y)$ where $E_{\theta^*(y)}$ is similar to the y-section of $T^*(x, x, \ldots)$. This can easily be done in every case, except $x = 1/3$, to show that $l^*(x, x, \ldots) = 0$. When $x = 1/3$, the mapping $y \longrightarrow \theta^*(y)$ is constant. In fact each y-section is similar to $[0, 1]$. The set $T^*(1/3, 1/3, \ldots)$ consists of the two triangles with vertices $(0, 1)$, $(1, 1)$, $(1/2, 1/2)$ and $(0, 0)$, $(1, 0)$, $(1/2, 1/2)$.

We admit that the method of proof outlined above is somewhat artificial and does not readily generalize to constructions involving more than three intervals. We hope that a direct proof of Proposition 2 can be found which will tell precisely for which x it is true that $l(x, x, \ldots) = 0$.

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