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A SUMMATION FORMULA AND SOME PROPERTIES OF EULERIAN FUNCTIONS[1]

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ABSTRACT. A product formula for Eulerian functions and a general summation method are given.

The q-Eulerian function $H_k(x|q_1, \ldots, q_k)$ may be defined symbolically by $H^k = x^{-1}\prod_{j=1}^k (1 + q_j H)$ if $k \geq 1$; in addition, $H_0 = 1$. Roselle [4, (3.2) and (3.9)] proved the following:

$$(1) \qquad xH_k(x \mid q_1, \ldots, q_k) = \sum_{n_1,\ldots,n_k=0}^{\infty} q_1^{n_1} \cdots q_k^{n_k} x^{-\max(n_1,\ldots,n_k)},$$

$$(2) \qquad H_k(x^{-1} \mid q_1^{-1}, \ldots, q_k^{-1}) = (-1)^k x q_1 \cdots q_k H_k(x \mid q_1, \ldots, q_k).$$

These formulas are useful in obtaining the following:

$$(3) \qquad \begin{aligned} &\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} (\min(n_1, \ldots, n_k))^m q_1^{n_1} \cdots q_k^{n_k} z^t \\ &\qquad = (1-z)^{-1}(-1)^{k+m} z H_k(z \mid q_1^{-1}, q_2^{-1}, \ldots, q_k^{-1}) H_m(q_1 q_2 \cdots q_k z). \end{aligned}$$

Proof of (3) is as follows. The left side becomes

$$\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \sum_{j=1}^{\min(n_1,\ldots,n_k)^m} q_1^{n_1} \cdots q_k^{n_k} z^t$$

$$= (1-z)^{-1} \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{1 \leq j \leq n_1^m,\ldots,n_k^m} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

$$= (1-z)^{-1} \sum_{j=1}^{\infty} \sum_{n_1,\ldots,n_k \geq j^{1/m}} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

$$= (1-z)^{-1} \sum_{j=1}^{\infty} \sum_{n_1,\ldots,n_k=0}^{\infty} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)} (q_1 \cdots q_k z)^{L(j^{1/m})}$$

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where $L(x)$ denotes the least integer $\geq x$. Using (1), then (2), and performing a little manipulation on the sum of index j, this last expression becomes

$$(1-z)^{-1}z^{-1}H_k(z^{-1}\mid q_1,\dots,q_k)\sum_{j=1}^{\infty}(q_1\cdots q_k z)^{L(j^{1/m})}$$

$$(4)\qquad = (1-z)^{-1}(-1)^k q_1^{-1}\cdots q_k^{-1}H_k(z\mid q_1^{-1},\dots,q_k^{-1})$$

$$\cdot\,(1-q_1q_2\cdots q_k)\sum_{j=0}^{\infty} j^m(q_1\cdots q_k z)^j.$$

Following Riordan [3, p. 38], the nth Eulerian polynomial $a_n(x)$ is defined by $a_n(x)=(1-x)^{n+1}\sum_{j\geq 0} j^n x^j$. Also $x(x-1)^k H_k(x)=a_k(x)$ where $H_k(x)$ is the ordinary Eulerian function $H_k(x\mid 1,1,\dots,1)$. The right side of (3) is obtained by making the substitution for the sum in (4).

If we put $q_1=q_2=\cdots=q_k=1$, then (3) becomes a new product formula for Eulerian functions (and hence polynomials):

$$(5)\qquad H_k(z)H_m(z)=(-1)^{k+m}\frac{1-z}{z}\sum_{t=0}^{\infty}\ \sum_{n_1,\dots,n_k=0}^{t}(\min(n_1,\dots,n_k))^m z^t.$$

Notice the curious symmetry in m and k.

Equation (5) is equivalent to

$$(6)\qquad \sum_{i+j=t}A_{k,i}A_{m,j}=\sum_{h=0}^{t}(-1)^h\binom{k+m+1}{h}\sum_{n_1,\dots,n_k=0}^{t-h-1}(\min(n_1,\dots,n_k))^m$$

where the $A_{n,j}$ are the Eulerian numbers defined by

$$(x-1)^n H_n(x)=\sum_{j=1}^{n}A_{n,j}x^{j-1}\qquad (n\geq 1).$$

Observe that (6) may be written as

$$\sum_{n_1,\dots,n_k=0}^{t-1}(\min(n_1,\dots,n_k))^m=\sum_{i+j+h=t}\binom{k+m+h}{h}A_{k,i}A_{m,j}$$

which includes, as a special case, Worpitzky's well-known result $x^k=\sum_{s=1}^{k}A_{k,s}\binom{x+s-1}{k}$. It also includes the content of a problem posed by R. L. Graham [1].

Let γ denote an increasing function on the real numbers such that $\gamma(0)=0$ and $\gamma(n)$ is a positive integer if n is a positive integer. Also let γ^{-} be the inverse function of γ. Then a generalization of (3) is

$$\sum_{t=0}^{\infty}\ \sum_{n_1,\dots,n_k=0}^{t}\ \sum_{j=0}^{\gamma(\min(n_1,\dots,n_k))} c_j q_1^{n_1}\cdots q_k^{n_k}z^t$$

$$(7)\qquad =(1-z)^{-1}(-1)^k q_1^{-1}\cdots q_k^{-1}H_k(z\mid q_1^{-1},\dots,q_k^{-1})\sum_{j=0}^{\infty}c_j(q_1\cdots q_k z)^{L(\gamma^{-}(j))}.$$

The proof of (3) carries over to (7), and if we put $c_0 = 0$, $c_j = 1$ $(j \geq 1)$ and $y(x) = x^n$, then (7) becomes (3). It may be of interest to compare (7) with the partition formulas of [2].

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