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A SUMMATION FORMULA AND SOME PROPERTIES OF EULERIAN FUNCTIONS[1]

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ABSTRACT. A product formula for Eulerian functions and a general summation method are given.

The q-Eulerian function $H_k(x|q_1, \ldots, q_k)$ may be defined symbolically by $H^k = x^{-1}\prod_{j=1}^k(1 + q_jH)$ if $k \geq 1$; in addition, $H_0 = 1$. Roselle [4, (3.2) and (3.9)] proved the following:

$$(1)\qquad xH_k(x \mid q_1, \ldots, q_k) = \sum_{n_1,\ldots,n_k=0}^{\infty} q_1^{n_1} \cdots q_k^{n_k} x^{-\max(n_1,\ldots,n_k)},$$

$$(2)\qquad H_k(x^{-1} \mid q_1^{-1}, \ldots, q_k^{-1}) = (-1)^k xq_1 \cdots q_k H_k(x \mid q_1, \ldots, q_k).$$

These formulas are useful in obtaining the following:

$$(3)\qquad \begin{aligned} &\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} (\min(n_1, \ldots, n_k))^m q_1^{n_1} \cdots q_k^{n_k} z^t \\ &= (1 - z)^{-1}(-1)^{k+m} z H_k(z \mid q_1^{-1}, q_2^{-1}, \ldots, q_k^{-1}) H_m(q_1q_2 \cdots q_k z). \end{aligned}$$

Proof of (3) is as follows. The left side becomes

$$\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \sum_{j=1}^{\min(n_1,\ldots,n_k)^m} q_1^{n_1} \cdots q_k^{n_k} z^t$$

$$= (1 - z)^{-1} \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{1\leq j\leq n_1^m,\ldots,n_k^m} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

$$= (1 - z)^{-1} \sum_{j=1}^{\infty} \sum_{n_1,\ldots,n_k\geq j^{1/m}} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

$$= (1 - z)^{-1} \sum_{j=1}^{\infty} \sum_{n_1,\ldots,n_k=0}^{\infty} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)} (q_1 \cdots q_k z)^{L(j^{1/m})}$$

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where $L(x)$ denotes the least integer $\geq x$. Using (1), then (2), and performing a little manipulation on the sum of index j, this last expression becomes

$$(1-z)^{-1}z^{-1}H_k(z^{-1}|q_1,\ldots,q_k)\sum_{j=1}^{\infty}(q_1\cdots q_k z)^{L(j^{1/m})}$$

$$(4) \qquad = (1-z)^{-1}(-1)^k q_1^{-1}\cdots q_k^{-1}H_k(z|q_1^{-1},\ldots,q_k^{-1})$$

$$\cdot(1-q_1q_2\cdots q_k)\sum_{j=0}^{\infty}j^m(q_1\cdots q_k z)^j.$$

Following Riordan [3, p. 38], the nth Eulerian polynomial $a_n(x)$ is defined by $a_n(x) = (1-x)^{n+1}\sum_{j\geq 0}j^n x^j$. Also $x(x-1)^k H_k(x) = a_k(x)$ where $H_k(x)$ is the ordinary Eulerian function $H_k(x|1, 1, \ldots, 1)$. The right side of (3) is obtained by making the substitution for the sum in (4).

If we put $q_1 = q_2 = \cdots = q_k = 1$, then (3) becomes a new product formula for Eulerian functions (and hence polynomials):

$$(5) \qquad H_k(z)H_m(z) = (-1)^{k+m}\frac{1-z}{z}\sum_{t=0}^{\infty}\sum_{n_1,\ldots,n_k=0}^{t}(\min(n_1,\ldots,n_k))^m z^t.$$

Notice the curious symmetry in m and k.

Equation (5) is equivalent to

$$(6) \qquad \sum_{i+j=t}A_{k,i}A_{m,j} = \sum_{h=0}^{t}(-1)^h\binom{k+m+1}{h}\sum_{n_1,\ldots,n_k=0}^{t-h-1}(\min(n_1,\ldots,n_k))^m$$

where the $A_{n,j}$ are the Eulerian numbers defined by

$$(x-1)^n H_n(x) = \sum_{j=1}^{n}A_{n,j}x^{j-1} \qquad (n \geq 1).$$

Observe that (6) may be written as

$$\sum_{n_1,\ldots,n_k=0}^{t-1}(\min(n_1,\ldots,n_k))^m = \sum_{i+j+h=t}\binom{k+m+h}{h}A_{k,i}A_{m,j}$$

which includes, as a special case, Worpitzky's well-known result $x^k = \sum_{s=1}^{k}A_{k,s}\binom{x+s-1}{k}$. It also includes the content of a problem posed by R. L. Graham [1].

Let γ denote an increasing function on the real numbers such that $\gamma(0) = 0$ and $\gamma(n)$ is a positive integer if n is a positive integer. Also let γ^- be the inverse function of γ. Then a generalization of (3) is

$$\sum_{t=0}^{\infty}\sum_{n_1,\ldots,n_k=0}^{t}\sum_{j=0}^{\gamma(\min(n_1,\ldots,n_k))}c_j q_1^{n_1}\cdots q_k^{n_k}z^t$$

$$(7) \qquad = (1-z)^{-1}(-1)^k q_1^{-1}\cdots q_k^{-1}H_k(z|q_1^{-1},\ldots,q_k^{-1})\sum_{j=0}^{\infty}c_j(q_1\cdots q_k z)^{L(\gamma^-(j))}.$$

The proof of (3) carries over to (7), and if we put $c_0 = 0$, $c_j = 1$ $(j \geq 1)$ and $y(x) = x^n$, then (7) becomes (3). It may be of interest to compare (7) with the partition formulas of [2].

REFERENCES

1. R. L. Graham, *Problem* E2507, Amer. Math. Monthly **81** (1974), 1111.

2. R. C. Grimson, *Some partition generating functions*, Proc. Fourth Southeastern Conf. on Combinatorics, Graph Theory, and Computing, 1973, pp. 299–308.

3. J. Riordan, *An introduction to combinatorial analysis*, Wiley Publ. in Math. Statist., Wiley, New York; Chapman & Hall, London, 1958. MR 20 #3077.

4. D. P. Roselle, *Generalized Eulerian functions and a problem in partitions*, Duke Math J. 33 (1966), 293–304. MR 32 #7532.

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