A SUMMATION FORMULA AND SOME PROPERTIES OF EULERIAN FUNCTIONS

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ABSTRACT. A product formula for Eulerian functions and a general summation method are given.

The q-Eulerian function $H_k(x|q_1,\ldots,q_k)$ may be defined symbolically by $H_k = H_{k-1}[x(1+q_1H)]$ if $k \geq 1$; in addition, $H_0 = 1$. Roselle [4, (3.2) and (3.9)] proved the following:

1. $xH_k(x|q_1,\ldots,q_k) = \sum_{n_1,\ldots,n_k=0}^{\infty} q_1^{n_1} \cdots q_k^{n_k} x^{\max(n_1,\ldots,n_k)},$

2. $H_k(x^{-1}|q_1^{-1},\ldots,q_k^{-1}) = (-1)^k x q_1^{-1} \cdots q_k^{-1} H_k(x|q_1,\ldots,q_k).$

These formulas are useful in obtaining the following:

$$\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \left(\min(n_1,\ldots,n_k)\right)^m q_1^{n_1} \cdots q_k^{n_k} z^t$$

$$= (1-z)^{-1} (-1)^{k+m} z^m H_m(z|q_1^{-1},q_2^{-1},\ldots,q_k^{-1}) H_m(q_1 z q_2 \cdots q_k z).$$

Proof of (3) is as follows. The left side becomes

$$\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \left(\min(n_1,\ldots,n_k)\right)^m q_1^{n_1} \cdots q_k^{n_k} z^t$$

$$= (1-z)^{-1} \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{1 \leq n_1,\ldots,n_k} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

$$= (1-z)^{-1} \sum_{j=1}^{\infty} \sum_{n_1,\ldots,n_k \geq j/m} q_1^{n_1} \cdots q_k^{n_k} z^{\max(n_1,\ldots,n_k)}$$

$$= (1-z)^{-1} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} q_1^{n_1} \cdots q_k^{n_k} \max(n_1,\ldots,n_k)(q_1 \cdots q_k z)^{L(j/m)}$$

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where \( L(x) \) denotes the least integer \( \geq x \). Using (1), then (2), and performing a little manipulation on the sum of index \( j \), this last expression becomes

\[
(1 - z)^{-1} z^{-1} H_k(z^{-1}|q_1, \ldots, q_k) \sum_{j=1}^{\infty} (q_1 \cdots q_k z)^L(j/m)
\]

\[
= (1 - z)^{-1} (-1)^k q_1^{-1} \cdots q_k^{-1} H_k(z|q_1^{-1}, \ldots, q_k^{-1})
\]

\[
\cdot (1 - q_1 q_2 \cdots q_k) \sum_{j=0}^{\infty} j^m (q_1 \cdots q_k z)^j.
\]

Following Riordan \([3, p. 38]\), the \( n \)th Eulerian polynomial \( a_n(x) \) is defined by \( a_n(x) = (1 - x)^n + \sum_{j \geq 0} j^nx^j \). Also \( x(x-1)^k H_k(x) = a_k(x) \) where \( H_k(x) \) is the ordinary Eulerian function \( H_k(x|1, 1, \ldots, 1) \). The right side of (3) is obtained by making the substitution for the sum in (4).

If we put \( q_1 = q_2 = \cdots = q_k = 1 \), then (3) becomes a new product formula for Eulerian functions (and hence polynomials):

\[
H_k(z) H_m(z) = (-1)^{k+m} \frac{1-z}{z} \sum_{t=0}^{\infty} \sum_{n_1, \ldots, n_k=0}^{t} (\min(n_1, \ldots, n_k))^m z^t.
\]

Notice the curious symmetry in \( m \) and \( k \).

Equation (5) is equivalent to

\[
\sum_{i+j=t} A_{k+i,m+j} = \sum_{h=0}^{t} (-1)^h \binom{k + m + 1}{h} \sum_{n_1, \ldots, n_k=0}^{t - h - 1} (\min(n_1, \ldots, n_k))^m
\]

where the \( A_{n,j} \) are the Eulerian numbers defined by

\[
(x-1)^n H_n(x) = \sum_{j=1}^{n} A_{n,j} x^{j-1} \quad (n \geq 1).
\]

Observe that (6) may be written as

\[
\sum_{n_1, \ldots, n_k=0}^{t - 1} (\min(n_1, \ldots, n_k))^m = \sum_{i+j+h=t} \binom{k + m + h}{h} A_{k+i,m+j}
\]

which includes, as a special case, Worpitzky's well-known result \( x^k = \sum_{s=1}^{k} A_{k,s} (x + s - 1) \). It also includes the content of a problem posed by R. L. Graham \([1]\).

Let \( \gamma \) denote an increasing function on the real numbers such that \( \gamma(0) = 0 \) and \( \gamma(n) \) is a positive integer if \( n \) is a positive integer. Also let \( \gamma^- \) be the inverse function of \( \gamma \). Then a generalization of (3) is

\[
\sum_{t=0}^{\infty} \sum_{n_1, \ldots, n_k=0}^{\gamma(\min(n_1, \ldots, n_k))} c_{j} q_1^{n_1} \cdots q_k^{n_k} z^t
\]

\[
= (1 - z)^{-1} (-1)^k q_1^{-1} \cdots q_k^{-1} H_j(z|q_1^{-1}, \ldots, q_k^{-1}) \sum_{j=0}^{\infty} c_{j} (q_1 \cdots q_k z)^{L(\gamma^-)(j)}.
\]
The proof of (3) carries over to (7), and if we put $c_0 = 0$, $c_j = 1 \ (j \geq 1)$ and $\gamma(x) = x^n$, then (7) becomes (3). It may be of interest to compare (7) with the partition formulas of [2].

REFERENCES


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