A BASIS RESULT FOR $\Sigma^0_3$ SETS OF REALS
WITH AN APPLICATION TO MINIMAL COVERS

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ABSTRACT. It is shown that every $\Sigma^0_3$ set of reals which contains reals of arbitrarily high Turing degree in the hyperarithmetic hierarchy contains reals of every Turing degree above the degree of Kleene's $\mathcal{O}$. As an application it is shown that every Turing degree above the Turing degree of Kleene's $\mathcal{O}$ is a minimal cover.

In this paper we consider a particular verification of the following nebulously stated and tenuously held principle: Every easily definable set of reals with enough complicated members contains members from any sufficiently large degree of complexity. Let $\mathcal{O}$ be the Turing degree of Kleene's $\mathcal{O}$. Our result is

**Theorem.** Any $\Sigma^0_3$ set of reals, with the property that every hyperarithmetic real is recursive in some member of it, contains reals of every Turing degree above $\mathcal{O}$.

As the first author noticed, an application of this theorem gives a new proof of Jockusch's result that there is a cone of minimal covers and estimates the base of that cone to be $\mathcal{O}$.

1. Preliminaries. Let $\omega = \{0, 1, 2, \ldots \}$ be the set of natural numbers and $R = \omega^\omega$ the set of all functions from $\omega$ to $\omega$ or (for simplicity) reals. Letters $i, j, k, l, m, \ldots$ will denote elements of $\omega$ and $\alpha, \beta, \gamma, \delta, \sigma, \tau, \ldots$ elements of $R$. We shall use, without explicit reference, standard facts of recursion theory, which can be found, for example, in [5] or [7].

The basic ingredient in the proof of our main theorem is the use of the well-known determinacy of closed games in its effective form (see, for example, [4]), which we proceed to state. For a general explanation of the connection between degrees of unsolvability and determinacy of games, see Martin [3]. Let $\langle a_1, \ldots, a_n \rangle$, where $a_i \in \omega$ or $a_i \in R$ be a trivial recursive coding of $n$-tuples by reals. For $A \subseteq R$ consider the game in which players I, II alternatively choose natural numbers $\alpha(0), \beta(0), \alpha(1), \beta(1), \ldots$ and I wins iff $(\alpha, \beta) \in A$. If $\sigma \in R$ is a strategy for player I, let $\sigma * \beta$ be the

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result of I's moves when he follows $\sigma$ against $\Pi$ playing $\beta$. More formally
$(\sigma \ast \beta)(n) = \sigma(\beta(n))$. Similarly let $\alpha \circ \tau$ be the result of $\Pi$'s moves when he
follows $\tau$ against $I$ playing $\alpha$, so that $(\alpha \circ \tau)(n) = \tau(\alpha(n + 1))$. Then we have

Fact (Determinacy of open games). Given a $\Pi^0_1$ set of reals $B$, either
(I) there is a real $\sigma$ recursive in $\mathcal{C}$ such that for all reals $\beta$, $(\sigma \ast \beta, \beta) 
\in B$, or

(II) there is a hyperarithmetic real $\tau$ such that for all reals $\alpha$, $(\alpha, \alpha \circ \tau) \notin B$.

2. Proof of the theorem. We are now ready to prove our main result.

Theorem. Any $\Sigma^0_3$ set of reals, with the property that every hyperarith-
metic real is recursive in some member of it, contains reals of every Turing
degree $\geq \emptyset$.

Proof. We first prove this for $\Pi^0_1$ sets of reals. Let thus $A \subseteq R$ be a
$\Pi^0_1$ set satisfying the hypotheses of the theorem. Let $\{e\}^\gamma$, where $e \in \omega$, $\gamma \in R$, denote the $e$th function from $\omega$ into $\omega$ partial recursive in $\gamma$ and put

\[ B = \{(e, \eta, \gamma, \beta) : e \in \omega \land \gamma \in \omega \land (\forall n)\{e\}^\gamma(n) \text{ converges in exactly } \eta(n) \text{ steps} \land \beta = (e, \eta, \gamma) \circ \{e\}^\gamma \}. \]

Clearly $B \subseteq R$ is $\Pi^0_1$, so by the fact in §1 either (I) or (II) holds. If (II) holds,
let $\tau$ be a hyperarithmetic real such that for all $\alpha \in R$, $(\alpha, \alpha \circ \tau) \notin B$. By
assumption, there is a $\gamma \in A$ such that $\tau$ is recursive in $\gamma$. So for some $e \in \omega$, $\{e\}^\gamma = \tau$. Let $\eta(n) = \text{number of steps in which } \{e\}^\gamma(n) \text{ converges}$. Let
$\alpha = (e, \eta, \gamma)$. Then $(\alpha, \alpha \circ \tau) \notin B$. But clearly $(\alpha, \alpha \circ \tau) \in B$, a contradic-
tion, so case (I) must hold. Thus there is $\sigma$ recursive in $\mathcal{C}$ such that for all $\beta$, $(\sigma \ast \beta, \beta) \in B$. Let $\mathcal{C}$ be recursive in $\beta$. We shall prove that a real of the same Turing degree as $\beta$ belongs to $A$. Since $(\sigma \ast \beta, \beta) \in B$, clearly
$\sigma \ast \beta = (e, \eta, \gamma)$ for some $e \in \omega$, $\eta, \gamma \in R$ such that $\gamma \in A$, $\{e\}^\gamma$ is total and $\eta(n) = \text{the number of steps in which } \{e\}^\gamma(n) \text{ converge}$. Also $\beta = (e, \eta, \gamma) \circ \{e\}^\gamma$. Thus $\beta$ is recursive in $\gamma$. But also $\sigma$ is recursive in $\beta$, thus $\gamma$ is recursive in $\beta$ i.e., $\gamma$ and $\beta$ have the same Turing degree. Since $\gamma \in A$, we are done.

The following observation will now complete the proof of the theorem:

For any $\Sigma^0_3$ set of reals $B$ there is a $\Pi^0_1$ set of reals $A$ such that for
all Turing degrees $d$, $d \cap B \neq \emptyset \iff d \cap A \neq \emptyset$.

To prove this, find a recursive set $R$ such that for all $\alpha$,

$\alpha \in A \iff \exists j \forall k R(i, j, \alpha(k))$

and let

$A = \{(i, \alpha, \beta) : \forall j(\beta(j) = \mu k R(i, j, \alpha(k)))\}$. Q.E.D.
Remarks. (1) The above theorem is best possible in the sense that it cannot be improved to include all $\Pi_3^0$ sets of reals—the set
\[ A = \{ a : (\forall e) (|e|^a \text{ is total} \rightarrow |e|^a \not\in F) \}, \]
where $F$ is a nonempty $\Pi_1^0$ set with no hyperarithmetic member, is a $\Pi_3^0$ set of reals which is closed under Turing reducibility and which contains all hyperarithmetic reals and yet which fails to contain Kleene’s $\mathcal{C}$.

(2) The proof of the theorem can be easily modified to show that
\[ \text{Determinacy}(\Sigma_n^0) \iff \text{Turing Determinacy}(\Sigma_{n+1}^0), \]
where for a collection of sets of reals $\Gamma$, Determinacy $(\Gamma) \iff \forall A \in \Gamma \ (A \text{ is determined})$ and Turing Determinacy $(\Gamma) \iff \forall A \in \Gamma \ (A \text{ is invariant under Turing equivalence} \Rightarrow A \text{ is determined})$. This result has been also proved independently (and a little earlier than us) by Ramez Sami (private communication). In fact the proof above shows that if every $\Sigma_n^0$ set is determined, then given a $\Sigma_{n+1}^0$ set of reals $A$ which is cofinal (i.e., for every $a \in \mathbb{R}$ there is $\beta \in A$ s.t. $a$ is recursive in $\beta$), there is a Turing degree $d$ such that $A$ contains reals of every degree $\geq d$. (To see this we note first that $A$ may be assumed to be $\Pi_1^0$ since if $a \in A \Rightarrow \exists mA^*(\langle m, a \rangle)$ where $A^* \in \Pi_0^0$, clearly for any degree $d$, $A \cap d \neq \emptyset \iff A^* \cap d \neq \emptyset$. Then we define $B$ exactly as in the proof of the theorem above and argue again that II cannot have a winning strategy in the game determined by $B$ since $B$ is cofinal. Then I has a winning strategy, since every $\Sigma_n^0$ game is determined, so by the argument given there, $A$ contains reals of every Turing degree above the degree of a winning strategy $\sigma$ for player I.) Using this fact, Martin [2] showed that
\[ \text{Determinacy}(\Sigma_n^0) \Rightarrow \text{Turing Determinacy}(\Delta_{n+2}^0) \]
which is best possible in analysis, since he also proved that Analysis $\not\Rightarrow$ Turing Determinacy($\Sigma_3^0$) (see [2]).

3. An application to minimal covers. Using $\Sigma_4^0$-determinacy, Jockusch [1] has shown that there is a cone of minimal covers. Harrington noticed that this result follows from our main theorem above (and hence follows from just open determinacy and so, in particular, is a theorem of analysis). This also allows for computing that Kleene’s $\mathcal{C}$ can be taken as a basis of the cone. Here is Harrington’s result.

**Theorem.** Every Turing degree $\geq \emptyset$ contains a minimal cover.

**Proof.** By Sacks [6], for every real $a$, there is a minimal cover of $a$ which is $\Delta_2^0$ in $a$, uniformly. Thus there is a $\Pi_2^0$ predicate $P(a, \beta)$ such that
\[ \forall a \forall \beta (P(a, \beta) \rightarrow (\beta \text{ is a minimal cover of } a)) \quad \text{and} \quad \forall a \exists ! \beta P(a, \beta). \]

Now $A = \{ (a, \beta) : P(a, \beta) \}$ is clearly a $\Pi_2^0$ set of reals with the property
that every hyperarithmetic real is recursive in some member of it. It is also
a collection of minimal covers. By the theorem in §2 every Turing degree
≥ Θ contains a member of A, thus every Turing degree ≥ Θ is a minimal
cover. Q.E.D.

We conclude with an open problem raised by Jockusch [1]. Is there a
hyperarithmetic real in a cone of minimal covers? In particular, is Θ^ω in
such a cone?

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