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A BASIS RESULT FOR Σ_3^0 SETS OF REALS WITH AN APPLICATION TO MINIMAL COVERS

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ABSTRACT. It is shown that every Σ_3^0 set of reals which contains reals of arbitrarily high Turing degree in the hyperarithmetic hierarchy contains reals of every Turing degree above the degree of Kleene's $\mathcal{O}$. As an application it is shown that every Turing degree above the Turing degree of Kleene's $\mathcal{O}$ is a minimal cover.

In this paper we consider a particular verification of the following nebulously stated and tenuously held principle: Every easily definable set of reals with enough complicated members contains members from any sufficiently large degree of complexity. Let $\mathcal{O}$ be the Turing degree of Kleene's $\mathcal{O}$. Our result is

Theorem. *Any Σ_3^0 set of reals, with the property that every hyperarithmetic real is recursive in some member of it, contains reals of every Turing degree above* $\mathcal{O}$.

As the first author noticed, an application of this theorem gives a new proof of Jockusch's result that there is a cone of minimal covers and estimates the base of that cone to be $\mathcal{O}$.

1. **Preliminaries.** Let $\omega = \{0, 1, 2, \ldots\}$ be the set of natural numbers and $R = {}^{\omega}\omega$ the set of all functions from ω to ω or (for simplicity) *reals*. Letters $i, j, k, l, m, \ldots$ will denote elements of ω and $\alpha, \beta, \gamma, \delta, \sigma, \tau, \ldots$ elements of R. We shall use, without explicit reference, standard facts of recursion theory, which can be found, for example, in [5] or [7].

The basic ingredient in the proof of our main theorem is the use of the well-known determinacy of closed games in its effective form (see, for example, [4]), which we proceed to state. For a general explanation of the connection between degrees of unsolvability and determinacy of games, see Martin [3]. Let $\langle a_1, \ldots, a_n\rangle$, where $a_i \in \omega$ or $a_i \in R$ be a trivial recursive coding of n-tuples by reals. For $A \subseteq R$ consider the game in which players I, II alternatively choose natural numbers $\alpha(0), \beta(0), \alpha(1), \beta(1), \ldots$ and I wins iff $\langle \alpha, \beta\rangle \in A$. If $\sigma \in R$ is a strategy for player I, let $\sigma * \beta$ be the

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result of I's moves when he follows σ against II playing β. More formally $(\sigma * \beta)(n) = \sigma(\overline{\beta}(n))$. Similarly let $\alpha \circ \tau$ be the result of II's moves when he follows τ against I playing α, so that $(\alpha \circ \tau)(n) = \tau(\overline{\alpha}(n+1))$. Then we have

Fact (Determinacy of open games). *Given a Π^0_1 set of reals B, either*

(I) *there is a real σ recursive in $\mathcal{O}$ such that for all reals β, $\langle \sigma * \beta, \beta\rangle \in B$, or*

(II) *there is a hyperarithmetic real τ such that for all reals α, $\langle \alpha, \alpha \circ \tau\rangle \notin B$.*

2. Proof of the theorem.

We are now ready to prove our main result.

Theorem. *Any Σ^0_3 set of reals, with the property that every hyperarithmetic real is recursive in some member of it, contains reals of every Turing degree $\geq \mathcal{O}$.*

Proof. We first prove this for Π^0_1 sets of reals. Let thus $A \subseteq R$ be a Π^0_1 set satisfying the hypotheses of the theorem. Let $\{e\}^\gamma$, where $e \in \omega$, $\gamma \in R$, denote the eth function from ω into ω partial recursive in γ and put

$$B = \{\langle\langle e, \eta, \gamma\rangle, \beta\rangle : e \in \omega \;\&\; \gamma \in A \;\&\; (\forall n)(\{e\}^\gamma(n) \text{ converges in exactly } \eta(n) \text{ steps}) \;\&\; \beta = \langle e, \eta, \gamma\rangle \circ \{e\}^\gamma\}.$$

Clearly $B \subseteq R$ is Π^0_1, so by the fact in §1 either (I) or (II) holds. If (II) holds, let τ be a hyperarithmetic real such that for all $\alpha \in R$, $\langle \alpha, \alpha \circ \tau\rangle \notin B$. By assumption, there is a $\gamma \in A$ such that τ is recursive in γ. So for some $e \in \omega$, $\{e\}^\gamma = \tau$. Let $\eta(n) =$ number of steps in which $\{e\}^\gamma(n)$ converges. Let $\alpha = \langle e, \eta, \gamma\rangle$. Then $\langle \alpha, \alpha \circ \tau\rangle \notin B$. But clearly $\langle \alpha, \alpha\circ\tau\rangle \in B$, a contradiction, so case (I) must hold. Thus there is σ recursive in $\mathcal{O}$ such that for all β, $\langle \sigma * \beta, \beta\rangle \in B$. Let $\mathcal{O}$ be recursive in β. We shall prove that a real of the same Turing degree as β belongs to A. Since $\langle \sigma * \beta, \beta\rangle \in B$, clearly $\sigma * \beta = \langle e, \eta, \gamma\rangle$ for some $e \in \omega$, $\eta, \gamma \in R$ such that $\gamma \in A$, $\{e\}^\gamma$ is total and $\eta(n) =$ the number of steps in which $\{e\}^\gamma(n)$ converges. Also $\beta = \langle e, \eta, \gamma\rangle \circ \{e\}^\gamma$. Thus β is recursive in γ. But also σ is recursive in β, thus γ is recursive in β i.e., γ and β have the same Turing degree. Since $\gamma \in A$, we are done.

The following observation will now complete the proof of the theorem:

For any Σ^0_3 set of reals B there is a Π^0_1 set of reals A such that for all Turing degrees d, $d \cap B \neq \emptyset \leftrightarrow d \cap A \neq \emptyset$.

To prove this, find a recursive set R such that for all α,

$$\alpha \in B \leftrightarrow \exists i \forall j \exists k R(i, j, \overline{\alpha}(k))$$

and let

$$A = \{\langle i, \alpha, \beta\rangle : \forall j(\beta(j) = \mu k R(i, j, \overline{\alpha}(k)))\}. \quad \text{Q.E.D.}$$

Remarks. (1) The above theorem is best possible in the sense that it cannot be improved to include all Π_3^0 sets of reals—the set

$$A = \{\alpha\colon (\forall e)(\{e\}^{\alpha} \text{ is total} \rightarrow \{e\}^{\alpha} \notin F)\},$$

where F is a nonempty Π_1^0 set with no hyperarithmetic member, is a Π_3^0 set of reals which is closed under Turing reducibility and which contains all hyperarithmetic reals and yet which fails to contain Kleene's $\mathcal{O}$.

(2) The proof of the theorem can be easily modified to show that Determinacy$(\Sigma_n^0) \Leftrightarrow$ Turing Determinacy(Σ_{n+1}^0), where for a collection of sets of reals Γ, Determinacy$(\Gamma) \Leftrightarrow \forall A \in \Gamma$ (A is determined) and Turing Determinacy $(\Gamma) \Leftrightarrow \forall A \in \Gamma$ (A is invariant under Turing equivalence $\Rightarrow A$ is determined). This result has been also proved independently (and a little earlier than us) by Ramez Sami (private communication). In fact the proof above shows that if every Σ_n^0 set is determined, then given a Σ_{n+1}^0 set of reals A which is cofinal (i.e., for every $\alpha \in R$ there is $\beta \in A$ s.t. α is recursive in β), there is a Turing degree d such that A contains reals of every degree $\geq d$. (To see this we note first that A may be assumed to be Π_n^0 since if $\alpha \in A \Leftrightarrow \exists m A^*(\langle m, \alpha\rangle)$ where $A^* \in \Pi_n^0$, clearly for any degree d, $A \cap d \neq \emptyset \Leftrightarrow A^* \cap d \neq \emptyset$. Then we define B exactly as in the proof of the theorem above and argue again that II cannot have a winning strategy in the game determined by B since B is cofinal. Then I has a winning strategy, since every Σ_n^0 game is determined, so by the argument given there, A contains reals of every Turing degree above the degree of a winning strategy σ for player I.) Using this fact, Martin [2] showed that Determinacy$(\Sigma_n^0) \Rightarrow$ Turing Determinacy(Δ_{n+2}^0) which is best possible in analysis, since he also proved that Analysis $\not\rightarrow$ Turing Determinacy(Σ_5^0) (see [2]).

3. **An application to minimal covers.** Using Σ_4^0-determinacy, Jockusch [1] has shown that there is a cone of minimal covers. Harrington noticed that this result follows from our main theorem above (and hence follows from just open determinacy and so, in particular, is a theorem of analysis). This also allows for computing that Kleene's $\mathcal{O}$ can be taken as a basis of the cone. Here is Harrington's result.

Theorem. *Every Turing degree $\geq \mathcal{O}$ contains a minimal cover.*

Proof. By Sacks [6], for every real α, there is a minimal cover of α which is Δ_2^0 in α, uniformly. Thus there is a Π_2^0 predicate $P(\alpha, \beta)$ such that

$$\forall\alpha\forall\beta(P\alpha, \beta) \rightarrow (\beta \text{ is a minimal cover of } \alpha) \quad \text{and} \quad \forall\alpha\exists!\beta P(\alpha, \beta).$$

Now $A = \{\langle\alpha, \beta\rangle\colon P(\alpha, \beta)\}$ is clearly a Π_2^0 set of reals with the property that every hyperarithmetic real is recursive in some member of it. It is also a collection of minimal covers By the theorem in §2 every Turing degree $\geq \mathcal{O}$ contains a member of A, thus every Turing degree $\geq \mathcal{O}$ is a minimal cover. Q.E.D.

We conclude with an open problem raised by Jockusch [1]. Is there a hyperarithmetic real in a cone of minimal covers? In particular, is $\mathcal{O}^{\omega}$ in such a cone?

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