

## JORDAN DERIVATIONS ON RINGS

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ABSTRACT. I. N. Herstein has shown that every Jordan derivation on a prime ring not of characteristic 2 is a derivation. This result is extended in this paper to the case of any ring in which  $2x = 0$  implies  $x = 0$  and which is semiprime or which has a commutator which is not a zero divisor.

1. **Introduction.** An additive mapping  $D$  of an (associative) ring into itself is a derivation if  $D(ab) = aDb + (Da)b$  for all elements  $a$  and  $b$  of the ring. For a ring  $R$  in which  $2x = 0$  implies  $x = 0$ , an additive mapping  $D$  of  $R$  into itself is said to be a *Jordan derivation* if

$$D(a \circ b) = a \circ Db + (Da) \circ b$$

for all  $a$  and  $b$  in  $R$ , where  $x \circ y = xy + yx$  is the Jordan product of  $x$  and  $y$  in  $R$ . Thus every derivation on  $R$  is a Jordan derivation, and the aim of this paper is to extend the class of rings for which it is known that the converse of this is true. This class contains all commutative rings in which  $2x = 0$  implies  $x = 0$  and all rings  $R$  with identity such that every Jordan homomorphism of  $R$  is the sum of a homomorphism and an antihomomorphism [4, Theorem 22]. I. N. Herstein [1, Theorem 3.1] showed that every Jordan derivation on a prime ring not of characteristic 2 is a derivation, and it is proved in [6, Theorem 3.3] that every continuous Jordan derivation on a semi-simple Banach algebra is a derivation. I shall extend these results to the case of any ring in which  $2x = 0$  implies  $x = 0$  and which is semiprime or which has a commutator which is not a zero divisor. In §3 I shall give some simple examples of Jordan derivations which are not derivations.

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2. Throughout this section  $D$  will denote a Jordan derivation on a ring  $R$  in which  $2x = 0$  implies  $x = 0$  and  $d$  will be the mapping from  $R^2 = \{(a, b) : a, b \in R\}$  to  $R$  defined by

$$d(a, b) = D(ab) - aDb - (Da)b.$$

The mapping  $d$  is additive with respect to pointwise addition on  $R^2$  and is zero if  $D$  is a derivation. I shall use the notation  $[a, b] = ab - ba$ ,  $[a, b, c] = abc + cba$

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The main results follow from the fact that  $[a, b] \circ d(a, b)$  and  $[[a, b], r, d(a, b)]$  are zero for all  $a, b$  and  $r$  in  $R$ . This can be proved directly, but is more easily obtained from the following analogous results for Jordan homomorphisms proved in [2, pp. 48–49]. An additive mapping  $J$  of  $R$  into a ring  $S$  (in which  $2x = 0$  implies  $x = 0$ ) is a Jordan homomorphism if  $J(a \circ b) = (Ja) \circ (Jb)$  for all  $a$  and  $b$  in  $R$ .

**Lemma 1.** *Let  $J$  be a Jordan homomorphism of  $R$  into a ring  $S$  in which  $2x = 0$  implies  $x = 0$ . Then, for all  $a, b$  and  $r$  in  $R$ ,*

$$(1) \quad (J(ab) - (Ja)(Jb))(J(ba) - (Ja)(Jb)) = 0,$$

$$(2) \quad [J(ab) - (Jb)(Ja), Jr, J(ab) - (Ja)(Jb)] = 0.$$

**Lemma 2.** *For all  $a, b$  and  $r$  in  $R$ ,*

$$(1) \quad d(a, b)[a, b] = 0 = [a, b]d(a, b),$$

$$(2) \quad [[a, b], r, d(a, b)] = 0.$$

**Proof.** Let  $S$  be the ring obtained from  $R^2$  by defining the product of  $(a, b)$  and  $(s, t)$  to be  $(as, at + bs)$ . Then the mapping  $J$  from  $R$  into  $S$ , defined by  $Ja = (a, Da)$ , is a Jordan homomorphism. By Lemma 1(1),

$$(0, d(a, b))([b, a], D(ba) - aDb - (Da)b) = 0,$$

and, therefore,  $d(a, b)[a, b] = 0$  for all  $a$  and  $b$  in  $R$ . Since  $D$  is also a Jordan derivation on the ring obtained from  $R$  by reversing the product, it follows that  $[a, b]d(a, b) = 0$ . Similarly, by Lemma 1(2),

$$([[a, b], D(ab) - bDa - (Db)a], (r, Dr), (0, d(a, b))) = 0,$$

and, therefore,  $[[a, b], r, d(a, b)] = 0$  for all  $a, b$  and  $r$  in  $R$ .

The next lemma is similar to [2, Lemma 3.10].

**Lemma 3.** *Let  $P$  be a prime ideal in  $R$  (i.e.  $aRb \subseteq P \rightarrow a \in P$  or  $b \in P$ ). Then  $[a, r, b] = 0$  for all  $r$  in  $R$  implies  $a \in P$  or  $b \in P$ .*

**Proof.** Let  $h$  and  $k$  be arbitrary elements of  $R$ . Then  $ahakk + bhaka = 0$  and, therefore,  $-2ahhbka = 0$ , since,  $akb + bka = 0 = bha + ahb$ . But then  $ahbRa \subseteq P$  for all  $h$  in  $R$ , and if  $a \notin P$ , then  $aRb \subseteq P$  and so  $b \in P$ .

The Baer lower radical, or prime radical,  $L$  of  $R$  is the intersection of all the prime ideals of  $R$ . For alternative definitions and properties of  $L$  see [5, pp. 69–71] or [3, pp. 193–197].  $R$  is said to be *semiprime* if  $L = (0)$ .

**Theorem 4.** *Let  $D$  be a Jordan derivation on a ring  $R$  in which  $2x = 0$  implies  $x = 0$ . Then for all  $a$  and  $b$  in  $R$ ,  $D(ab) - aDb - (Da)b$  is in  $L$ , the*

*Baer lower radical of  $R$ .*

**Proof.** First let  $P$  be a prime ideal such that the quotient ring  $R/P$  is not commutative. By Lemma 2(2),  $[[a, b], r, d(a, b)] = 0$  for all  $r$  in  $R$  and so, by Lemma 3,  $[a, b] \notin P$  implies  $d(a, b) \in P$ . Now suppose  $[a, b] \in P$  and that there exists an element  $r$  in  $R$  such that  $[a, r] \notin P$ . Then  $[a, b + r] \notin P$  and so both  $d(a, r)$  and  $d(a, b + r)$  are in  $P$ . But then  $d(a, b) = d(a, b + r) - d(a, r) \in P$ . Finally, if  $[a, r] \in P$  for all  $r$  in  $R$  and  $[s, r] \notin P$ , then  $[a + s, r] \notin P$  implies  $d(a + s, b) \in P$ . But  $[s, r] \notin P$  implies  $d(s, b) \in P$  and so, again,  $d(a, b) = d(a + s, b) - d(s, b) \in P$ .

If  $Q$  is a prime ideal such that  $R/Q$  is commutative, then  $R/Q$  is an integral domain. By Lemma 2(1),  $d(a, b) \circ [a, b] = 0$ . Thus, since  $D$  is a Jordan derivation,

$$d(a, b) \circ D([a, b]) + D(d(a, b)) \circ [a, b] = 0,$$

and, therefore,  $2d(a, b)D([a, b]) \in Q$ . It follows that either  $2d(a, b) \in Q$  or  $D([a, b]) \in Q$ . However,

$$2d(a, b) - D([a, b]) = D(a \circ b) - 2(aDb + (Da)b) \in Q,$$

so that, in either case,  $2d(a, b) \in Q$ . Since  $L$  is the intersection of all the prime ideals of  $R$ , we have  $2d(a, b) \in L$ . But  $L$  is a nil ideal [5, Theorem 4.21], and so there exists a positive integer  $n$  such that  $2^n(d(a, b))^n = 0$ . The condition  $2x = 0$  implies  $x = 0$  shows that  $d(a, b)$  is nilpotent and, therefore, contained in every prime ideal  $Q$  such that  $R/Q$  is commutative. Thus  $d(a, b) \in L$  for all  $a$  and  $b$  in  $R$ .

**Corollary 5.** *If  $R$  is semiprime then  $D$  is a derivation.*

**Theorem 6.** *Let  $R$  be a ring where  $2x = 0$  implies  $x = 0$  and which has a commutator which is not a zero divisor. Then every Jordan derivation on  $R$  is a derivation.*

**Proof.** By Lemma 2(2), for all  $a, r, s$ , and  $t$  in  $R$ ,  $[[s + a, t], r, d(s + a, t)] = 0$ . It follows that

$$[[s, t], r, d(a, t)] + [[a, t], r, d(s, t)] = 0,$$

and, on replacing  $t$  by  $t + b$ , that

$$[[s, t], r, d(a, b)] + [[s, b], r, d(a, t)] + [[a, t], r, d(s, b)] + [[a, b], r, d(s, t)] = 0$$

for all  $a, b, r, s$  and  $t$  in  $R$ . Let  $s$  and  $t$  be elements of  $R$  such that  $[s, t]z = 0$  or  $z[s, t] = 0$  implies  $z = 0$ . Then by Lemma 2(1),  $d(s, t) = 0$  and so the substitutions  $a = s$  and  $b = t$  give, respectively,  $[[s, t], r, d(s, b)] = 0$  and  $[[s, t], r, d(a, t)] = 0$ . But then, as in the proof of Lemma 3 (with  $h = k = [s, t]$ ),  $[s, t]^2 d(s, b)[s, t]^2 = 0$  and  $[s, t]^2 d(a, t)[s, t]^2 = 0$  and so  $d(s, b) = 0$

and  $d(a, t) = 0$  for all  $a$  and  $b$  in  $R$ . Finally,  $[[s, t], r, d(a, b)] = 0$  and, therefore, by the same argument,  $d(a, b) = 0$  for all  $a$  and  $b$  in  $R$ , and  $D$  is a derivation.

3. **Jordan derivations which are not derivations.** Let  $R$  be a ring in which  $2x = 0$  implies  $x = 0$  and such that either

(1)  $R$  has an element  $x$  such that  $axa = 0$  for all  $a$  in  $R$ , but  $axb \neq 0$  for some  $a$  and  $b$ , or

(2)  $a^2 = 0$  for all  $a$  in  $R$ , but  $ab \neq 0$  for some  $a$  and  $b$ .

In the first case the mapping  $Da = ax$  is a Jordan derivation, since

$$(a \circ b)x - a \circ (bx) - ax \circ b = -(bxa + axb) = -(a + b)x(a + b) = 0,$$

but not a derivation since,  $(ab)x - a(bx) - (ax)b = -axb \neq 0$ .

In the second case every additive mapping of  $R$  into itself is a Jordan derivation, since the Jordan product is zero. But the identity mapping, for example, is not a derivation. A ring of the second type is necessarily nilpotent ( $abc = 0$  for all  $a, b$ , and  $c$  in  $R$ ). An example is the Banach algebra obtained from  $\mathbf{R}^3$  (with the Euclidean norm) by defining the product of  $\mathbf{a}$  and  $\mathbf{b}$  in  $\mathbf{R}^3$  to be their vector product,  $\mathbf{a} \times \mathbf{b}$ , projected onto one of the coordinate axes. The problem remains of finding reasonable necessary conditions that  $R$  has Jordan derivations which are not derivations.

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