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ON PARACOMPACT SUBSETS OF LINEAR TOPOLOGICAL SPACES

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ABSTRACT. It is shown that a connected open subset of a σ-compact topological space is paracompact in the relative topology only if the subspace is σ-compact. An application is made to demonstrate the existence of nonparacompact open subspaces, in the weak-star and bounded weak-star topologies, of the dual to a nonseparable Banach space. As a corollary, nonempty paracompact manifolds modeled on such a space always have open submanifolds which are not paracompact.

It is sometimes necessary in analysis to consider function spaces which are differentiable manifolds modeled on locally convex spaces other than Banach spaces. For instance, let M_1 and M_2 be C^∞ manifolds, M_1 compact and M_2 finite-dimensional. Then examples of such function spaces are the following:

(1) $C^\infty(M_1, M_2)$, the space of smooth maps from M_1 to M_2, which is a differentiable manifold modeled on a Fréchet nuclear space.

(2) $L^p_k(M_1, M_2)$, where $k > (\text{dimension}(M_1))/p$, the space of Sobolev maps from M_1 to M_2 whose distributional derivatives are in L^p for those derivatives of order less than or equal to k.

Of course, it has been known for some time that example (2) is a differentiable Banach manifold. More recently, the author observed in [2] that $L^p_k(M_1, M_2)$ also possesses the structure of a differentiable bw^* manifold, and it will be shown in [3] that this bw^* manifold structure plays a role in that segment of the calculus of variations known as Morse-Palais-Smale theory.

Now, the class of bw^* spaces may be divided naturally into two subclasses. It was shown in [2] that all of the function space bw^* manifolds which are currently known are manifolds modeled on bw^* spaces lying in one of the classes (the so-called Bw^* spaces). It is relatively easy to develop a highly structured abstract theory for differentiable manifolds modeled on Bw^* spaces, and in [2] the author asked whether this theory could also be developed for manifolds modeled on bw^* spaces of the other type.

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It is the purpose of this article to answer that question in the negative: more specifically, if X is a bw^* space of the second type, and M is a paracompact topological manifold modeled on X, the M contains open submanifolds which are not paracompact. In fact, these open submanifolds can be chosen to lie inside of coordinate neighborhoods, so what we will really prove is a result about open subsets of linear topological spaces.

The first part of this note consists of results of a purely point-set topological nature. The ideas for the statement and proof of the basic result of this section (Theorem 1) come from a similar though slightly weaker result in [4].

Lemma 1. *Let Z be a connected topological space and let $\mathfrak{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an open cover of Z which has the property that, for each $\lambda_0 \in \Lambda$, $U_{\lambda_0} \cap U_\lambda$ is nonempty for at most a countable number of the U_λ. Then $\Lambda_0 = \{\lambda \in \Lambda \colon U_\lambda \neq \phi\}$ is countable.*

Proof. Assume Z is nonempty. Choose $\lambda_1 \in \Lambda$ such that $U_{\lambda_1} \neq \phi$, and define $\Lambda_1 = \{\lambda_1\}$. We will now define an increasing sequence $\{\Lambda_n \colon n \in N\}$ of subsets of Λ by induction: let $n > 1$, and assume that Λ_{n-1} has already been defined. Let $\Lambda_n = \{\lambda \in \Lambda \colon \exists \nu \in \Lambda_{n-1} \ni U_\lambda \cap U_\nu \neq \emptyset\}$.

I claim that each Λ_n is a countable set. We verify this by induction: it is certainly true if $n = 1$. Let $n > 1$, and assume that Λ_{n-1} is countable. Then, since $\Lambda_n = \bigcup_{\nu \in \Lambda_{n-1}} \{\lambda \colon U_\lambda \cap U_\nu \neq \emptyset\}$, Λ_n is the union of countable number of sets, each of which is itself countable. Thus Λ_n is countable.

Let $\Lambda_0 = \bigcup_{n \in N} \Lambda_n$. Then Λ_0 is certainly a countable subset of Λ, and U_λ is nonempty for each $\lambda \in \Lambda_0$. To see that $\Lambda_0 = \{\lambda \colon U_\lambda \neq \phi\}$, we proceed as follows: for each $n \in N$, let $W_n = \bigcup_{\lambda \in \Lambda_n} U_\lambda$, and let $W = \bigcup_{n \in N} W_n$. It is easy to see that $W = Z$. We will show this by showing that $W = \overline{W}$. Then, since W is nonempty and open, and since Z is connected, it will follow that $W = Z$. So let x be a limit point of W, and let λ_x be an element of Λ such that $x \in U_{\lambda_x}$. Since x is a limit point of W, $U_{\lambda_x} \cap W \neq \phi$, which implies there exists $n_x \in N$ such that $U_{\lambda_x} \cap W_{n_x} \neq \phi$. This in turn implies the existence of $\nu_x \in \Lambda_{n_x}$ such that $U_{\lambda_x} \cap U_{\nu_x} \neq \phi$, which implies that $\lambda_x \in \Lambda_{n_x+1}$. Thus $U_{\lambda_x} \subset W_{n_x+1} \subset W$, and we conclude that $x \in W$. Thus W contains all of its limit points, and so $W = \overline{W}$. Thus $W = Z$.

Now, let λ_0 be any element of Λ such that $U_{\lambda_0} \neq \phi$. Since $\bigcup_{n \in N} W_n = Z$, there is some $n_0 \in N$ such that $U_{\lambda_0} \cap W_{n_0} \neq \phi$, which implies that $\lambda_0 \in \Lambda_{n_0+1}$. Thus $\lambda_0 \in \Lambda_0$, and so we have proved that $\Lambda_0 = \{\lambda \in \Lambda \colon U_\lambda \neq \phi\}$.

Theorem 1. *Let Z be a connected paracompact topological space. Assume that, for each $x \in Z$, there is a neighborhood A_x of x (A_x need not necessarily be open) such that A_x is a Lindelöf space in the relative topology. Then Z is Lindelöf.*

Proof. For each $x \in Z$, let A_x be a Lindelöf neighborhood of x, and let W_x be the interior of A_x, so that $\mathfrak{W} = \{W_x: x \in Z\}$ is an open cover of Z. Since Z is paracompact, there is a locally finite open cover $\mathfrak{U} = \{U_\lambda: \lambda \in \Lambda\}$ which refines $\mathfrak{W}$. For each $x \in Z$, let V_x be an open neighborhood of x which intersects only a finite number of elements of $\mathfrak{U}$, and let $\mathfrak{V} = \{V_x: x \in X\}$.

Let $\lambda \in \Lambda$, and choose a point x_λ such that $U_\lambda \subset A_{x_\lambda}$. Since A_{x_λ} is Lindelöf, there is a sequence $V_1, V_2, V_3, \ldots$ of sets in $\mathfrak{V}$ which cover A_{x_λ}. Since each V_i intersects only a finite number of sets in $\mathfrak{U}$, $\bigcup_{i \in N} V_i$ intersects at most a countable number of sets in $\mathfrak{U}$. But $U_\lambda \subset \bigcup_{i \in N} V_i$, which implies that U_λ intersects at most a countable number of sets in $\mathfrak{U}$. Since this is true for each $\lambda \in \Lambda$, Lemma 1 implies that $\Lambda' = \{\lambda: U_\lambda \neq \phi\}$ is countable. Thus $\{A_{x_\lambda}: \lambda \in \Lambda'\}$ is a countable collection of Lindelöf subspaces of Z which covers Z, which implies that Z is Lindelöf.

Corollary 1. *Let Z be a Lindelöf regular topological space, U a connected open subset of Z. Then U is paracompact $\Longleftrightarrow$ U is Lindelöf.* □

Proof. Assume that U is Lindelöf. Since subspaces of regular spaces are regular, U is regular and Lindelöf, and hence is paracompact by a well-known result (see, for example, [5]). Conversely, assume that U is paracompact. Since Z is regular, there exists, for each $x \in U$, an open set $W_x \subset Z$ such that $x \in W_x \subset \overline{W_x} \subset U$. Since $\overline{W_x}$ is a closed subset of Z, it is Lindelöf. Thus Theorem 1 implies that U is Lindelöf. □

Theorem 2. *Let Z be a σ-compact regular topological space, U a connected open subset of Z. Then the following statements are equivalent*:

(1) *U is paracompact.*

(2) *U is σ-compact.*

(3) *There exists a continuous function $f: Z \to \mathbf{R}$ such that $f^{-1}(0) = Z - U$.*

(4) *$Z - U$ is a G_δ in Z.*

Proof. The equivalence of (2), (3) and (4) is well known for open subsets of Z (whether they are connected or not). Similarly, if U is σ-compact, then it is Lindelöf, and hence paracompact by Corollary 2. So the only thing to prove here is that (1) implies (2).

Assume that U is paracompact. Then U is Lindelöf, by Corollary 2. For each $x \in U$, let W_x be a neighborhood of x in U such that $x \in W_x \subset \overline{W_x} \subset U$. Note that $\{W_x: w \in U\}$ is an open cover of U. Since U is Lindelöf, there is a sequence $x_1, x_2, x_3, \ldots$ of points of U such that $\bigcup_{n \in N} W_{x_n} = U$. Thus $\{\overline{W_{x_n}}: n \in N\}$ is a countable collection of subspaces of U which covers U. Since each $\overline{W_{x_n}}$ is closed in Z, it is σ-compact. Thus U is σ-compact. □

It will be useful to mention one more standard result:

Lemma 2. *Let Z be a σ-compact regular topological space, U an open subset of Z. Assume that compact subsets of Z are metrizable. Then U is σ-compact.*

Proof. Let $\{C_n: n \in N\}$ be a sequence of compact subsets of Z which covers Z. Then $U = \bigcup_{n \in N}(U \cap C_n)$. It is an elementary result that open subsets of compact metric spaces are σ-compact. Thus U is σ-compact.

The value of Theorem 2 lies in the fact that it gives us relatively simple criteria for determining whether or not connected open subsets of σ-compact spaces are paracompact in the subspace topology. Since we are looking for open subsets which are not paracompact, Lemma 2 tells us that we must restrict our search to σ-compact spaces which have nonmetrizable compact subsets. An example of what we can show is the following:

Example. Consider the space $I^{\aleph_1}$, the product of the closed unit interval with itself $\aleph_1$ times. This is compact Hausdorff. Let x be a point in $I^{\aleph_1}$, and let $U = I^{\aleph_1} - \{x\}$. Then U is arcwise connected, and hence connected. Now it is easy to see that $\{x\}$ cannot be a G_δ in $I^{\aleph_1}$, which implies that U is not paracompact.

We now turn our attention to bw^* spaces. Although there exist intrinsic definitions of these linear topological spaces, we will for the sake of brevity restrict ourselves to a nonintrinsic definition which has the advantage of being more familiar to most readers.

Definition. Let V be a Banach space, E the dual space of V. Denote by X the underlying vector space of E, endowed with the strongest topology which agrees with the weak-star (w^*) topology on bounded subsets of E. We will call such a space X a bw^* space.

Note that the underlying sets of E and X are the same, so that a subset of X may be simultaneously considered as contained in both E and X.

Proposition. (1) *X is a locally convex space.*

(2) *X is σ-compact.*

(3) *X is regular.*

(4) *The strong dual of X is V.*

Proof. Two very different proofs of (1) may be found in [1, p. 428] and in [2, p. 45]. (2) is a consequence of the fact that the closed unit ball of E is compact in the w^* topology. (3) is true since any T_0 linear topological space is automatically Hausdorff and regular. (4) is a classic result due to Banach. Different versions of proof of (4) may be found in [6, p. 112] or in [2, p. 48]. □

Note that $X'' = E$, and that the canonical injective mapping of X into

X'' is just the identity map on the underlying sets of X and E. In particular, this map is onto.

Definition. A Bw^* space is a bw^* space which has a separable strong dual.

Lemma 3. *Let X be a bw^* space. Then X is a Bw^* space $\Leftrightarrow$ compact subsets of X are metrizable.*

Proof. See [1, p. 426]. □

Thus all open subsets of Bw^* spaces are paracompact. We will now show that, if X is a bw^* space with a nonseparable dual space, then X always has open subspaces which are not paracompact. But, in order to proceed, we will need a specific set of seminorms on X: let X be a bw^* space, and let $M = \{s\colon s = \{l_i\}_{i \in N}$ is a sequence in X' which converges to $0\}$. For each $s \in M$, define a seminorm λ_s on X by $\lambda_s(v) = \sup_{i \in N}\{|l_i(v)|\colon i \in N\}$ for each $v \in X$.

Lemma 4. *λ_s is a continuous seminorm on X for each $s \in M$. Furthermore, for each open neighborhood U of 0 in X, there is a sequence $s = s(U) \in M$ such that $\lambda_s^{-1}([0, 1)) \subset U$.*

Proof. See [1, p. 427]. □

Lemma 5. *Let X be a bw^* space, and assume that X' is not separable. Let A be a subspace of X such that:*

(1) $0 \in A$.

(2) A is a G_δ.

Then there exists a linear subspace Z of X such that dimension$(Z) > 0$ *and $Z \subset A$.*

Proof Let $U_1, U_2, \ldots$ be a sequence of open subsets of Z such that $A = \bigcap_{n \in N} U_n$. For each $n \in N$, choose a sequence $s_n = \{l_{in}\}_{i \in N}$ in M such that $\lambda_{s_n}^{-1}([0, 1)) \subset U_n$.

Let Y be the closed linear subspace of X' spanned by $\{l_{in}\colon i, n \in N\}$. Since Y is separable, and X' is nonseparable, there exist vectors in X' which do not lie in the subspace Y. Let X be such a vector. Then $x \neq 0$, so the Hahn-Banach theorem implies there exists $f \in X''$ such that $f(x) \neq 0$, and $f(y) = 0$ for all $y \in Y$.

Since X is semireflexive, f is contained in X. So let Z be the one-dimensional subspace of X spanned by f. Then Z has the required properties. □

Lemma 6. *Let X be a bw^* space which has a nonseparable strong dual. Let U be a nonempty open subset of X, and assume that $x \in U$. Then $U - \{x\}$ is not paracompact in the subspace topology.*

Proof. It suffices to make the additional assumptions that U is connected and $x = 0$. Thus $U - \{0\}$ is connected. However, any nonzero linear subspace of X must intersect $U - \{0\}$, so that Lemma 5 implies that $U' \cup \{0\}$ is not a G_δ in X. Thus $U - \{0\}$ is not paracompact. □

Theorem 3. *Let X be a bw^* space which has a nonseparable strong dual. Let U be a nonempty open subset of X, and assume that $x \in U$. Then there exist open neighborhoods U_1 and U_2 of X such that:*

(1) $U_i \subset U$, $i = 1, 2$.

(2) U_1 *is not paracompact.*

(3) U_2 *is paracompact.*

Proof. Let y be a point in U such that $y \neq x$, and let $U_1 = U - \{y\}$. Then Lemma 6 implies that U_1 is not paracompact.

To show the existence of U_2, assume that $x = 0$, and let λ be a continuous seminorm on X such that $\lambda^{-1}([0, 1)) \subset U$. Let $K(n)$ be the closed ball around 0 of radius n in $E = X''$, so that $K(n)$ is a compact subset of X. For each $n \in N$, let $C_n = \lambda^{-1}([0, 1 - 1/n]) \cap K(n)$. Then each C_n is compact in X, and $\lambda^{-1}([0, 1)) = \bigcup_{n \in N} C_n$. So let $U_2 = \lambda^{-1}([0, 1))$. □

Thus, if X is a bw^* space which has a nonseparable strong dual, then X is both "locally paracompact", and "locally nonparacompact".

Note that very slight modifications of the techniques employed above will prove the corresponding results for the w^* topology on dual Banach spaces.

Corollary. *Let X be a bw^* space which has a nonseparable strong dual, and let M be a nonempty paracompact manifold modeled on X. Then there exist open submanifolds of M which are not paracompact.*

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