PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

# A CHANGE OF RINGS THEOREM AND THE ARTIN-REES PROPERTY

M. BORATYŃSKI

ABSTRACT. A two-sided ideal $\mathfrak{A}$ of the ring $R$ is said to have the left $AR$ property if for every left ideal $I$ and every $k$ there exists an $n$ such that $\mathfrak{A}^n \cap I \subset \mathfrak{A}^k I$. Let $R$ be a left noetherian ring and $\mathfrak{A}$ a two-sided ideal contained in its Jacobson radical. If $\mathfrak{A}$ has the AR property then $\operatorname{l\,gl\,dim} R \leq \operatorname{p\,dim} R/\mathfrak{A} + \operatorname{l\,gl\,dim} R/\mathfrak{A}$, where $\operatorname{p\,dim} R/\mathfrak{A}$ denotes the (left) projective dimension of the module $R/\mathfrak{A}$.

Let $R$ be a left noetherian ring and $J$ its Jacobson radical. Small has proved in [7] that $\operatorname{l\,gl\,dim} R \leq \operatorname{r\,w\,dim} R/\mathfrak{A} + \operatorname{l\,gl\,dim} R/\mathfrak{A}$, where $\mathfrak{A}$ is a twosided ideal contained in $J$. The natural question arises whether in the above formula right weak dimension can be replaced by left projective dimension. Fields [4] has produced an example which shows that the answer, in general, is "no". However we are able to prove that in case $\mathfrak{A}$ satisfies the AR property the answer is "yes".

Our result is derived from three lemmas. The first is a homological characterization of the AR property which may be of some independent interest and the third is an improvement of Small's result.

**Definition.** A two-sided ideal $\mathfrak{A}$ of the ring $R$ is said to have the (left) AR property if for every left ideal $I$ and every $k$ there exists an $n$ such that $\mathfrak{A}^n \cap I \subset \mathfrak{A}^k I$.

**Remark.** In the commutative noetherian case it follows from the Artin-Rees lemma that every ideal has the AR property.

In the sequel, $E(M)$ will denote an injective hull of the left module $M$.

**Lemma 1.** *Let $R$ be a left noetherian ring and $\mathfrak{A}$ its arbitrary two-sided ideal. Then the following two conditions are equivalent.*

1°. *$\mathfrak{A}$ has the AR property.*

2°. *$E(M) = \bigcup_i (0\colon \mathfrak{A}^i)$ for any left module $M$ whose every element is annihilated by some power of $\mathfrak{A}$ where* $(0\colon \mathfrak{A}^i) = \{x \in E(M) \mid \mathfrak{A}^i x = 0\}$.

**Proof.** 1° → 2°. We put $E_1 = \bigcup_i (0\colon \mathfrak{A}^i)$. It will be sufficient to prove that $E_1$ is injective because the latter contains $M$. Let $f\colon I \to E_1$ where

Received by the editors September 19, 1974 and, in revised form, October 7, 1974.
*AMS (MOS) subject classifications* (1970). Primary 16A60.
*Key words and phrases.* AR property, projective dimension, Jacobson radical.

$I$ is an arbitrary left ideal of the ring $R$. $f(I) \subset (0: \mathfrak{A}^k)$ for some $k$ because $I$ is finitely generated. We pick $n$ such that $\mathfrak{A}^n \cap I \subset \mathfrak{A}^k I$. $\mathfrak{A}^n \cap I$ is contained in the kernel of $f$ because $f(\mathfrak{A}^n \cap I) \subset f(\mathfrak{A}^k I) = \mathfrak{A}^k f(I) = 0$.

We have the diagram of $R/\mathfrak{A}^n$-modules with an exact row:

$$\begin{tikzcd} 0 \arrow[r] & I/\mathfrak{A}^n \cap I \arrow[r] \arrow[d, "\bar{f}"'] & R/\mathfrak{A}^n \\ & (0: \mathfrak{A}^n) & \end{tikzcd}$$

$\bar{f}$ is the homomorphism induced by $f$ (we can assume that $n > k$). $(0: \mathfrak{A}^n)$ is an injective $R/\mathfrak{A}^n$-module. So there exists a homomorphism $R/\mathfrak{A}^n \to (0: \mathfrak{A}^n)$ which makes the above diagram commutative. This homomorphism composed with the natural homomorphism $R \to R/\mathfrak{A}^n$ is the desired extension of $f$ to $R$.

$2° \to 1°$. Let us consider the natural homomorphism $f: I \to I/\mathfrak{A}^k I \subset E(I/\mathfrak{A}^k I)$. There exists an element $x \in E(I/\mathfrak{A}^k I)$ such that $f(r) = rx$ for all $r \in I$. By our assumption $x \in (0: \mathfrak{A}^n)$ for some $n$. It follows that $\mathfrak{A}^n \cap I$ is contained in the kernel of $f$, which coincides with $\mathfrak{A}^k I$, which was to be proved. Several cases of the above lemma are known [3], [5], [6].

In the sequel inj dim $M$, p dim $M$ will denote the injective and the projective dimension of the left module $M$ respectively.

**Lemma 2.** *Let $R$ be a left noetherian ring and $\mathfrak{A}$ its two-sided ideal which has the $AR$ property. If $M$ is a module whose every element is annihilated by some power of $\mathfrak{A}$ then* inj dim $M \leq$ sup p dim $S$, *where supremum is taken over the category of modules which are annihilated by $\mathfrak{A}$.*

**Proof.** It will suffice to prove that if $\operatorname{Ext}^{n+1}(S, M) = 0$ for all modules $S$ which are annihilated by $\mathfrak{A}$, then inj dim $M \leq n$. We start with $n = 0$. If $E(M)/M \neq 0$, then, by Lemma 1, this module contains a nonzero submodule $M_1/M$ which is annihilated by $\mathfrak{A}$. We have the exact sequence $0 \to M \to M_1 \to M_1/M \to 0$ which is nonsplitting because $E(M)$ is an essential extension of $M$ (i.e. every nonzero submodule of $E(M)$ has a nonzero intersection with $M$). This exact sequence represents a nonzero element of $\operatorname{Ext}^1(M_1/M, M)$ which contradicts our hypothesis.

Let $n > 0$. We consider the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$, $\operatorname{Ext}^n(S, E(M)/M) \cong \operatorname{Ext}^{n+1}(S, M) = 0$. By Lemma 1, each element of $E(M)/M$ is annihilated by some power of $\mathfrak{A}$. So by the inductive hypothesis, inj dim $E(M)/M \leq n - 1$. We infer that inj dim $M \leq n$, which was to be proved.

**Lemma 3.** *Let $R$ be a left noetherian ring and $J$ its Jacobson radical. Then* l gl dim $R$ = sup inj dim $S$, *where supremum is taken over the category of modules which are annihilated by $J$.*

**Proof.** Let $M$ be a finitely generated module. We shall prove that the condition $\operatorname{Ext}^{n+1}(M, S) = 0$ for all modules $S$ which are annihilated by $J$

implies that $\mathrm{p\ dim}\ M \leq n$. Let $n = 0$. We have an exact sequence $0 \to K \xrightarrow{\alpha} F \to M \to 0$, where $F$ is a finitely generated free module. The map induced by $\alpha$, $\mathrm{Hom}(F, K/JK) \to \mathrm{Hom}(K, K/JK)$, is an epimorphism because $\mathrm{Ext}^1(M, K/JK) = 0$. In particular the natural homomorphism $K \to K/JK$ which will be denoted by $\gamma$ can be extended to the homomorphism $\delta\colon F \to K/JK$. $F$ is a free module, so there exist $\beta\colon F \to K$ such that $\gamma\beta = \delta$. We obtain that $\gamma\beta\alpha = \gamma$. Let us put $\epsilon = \beta\alpha - 1$. Then $\epsilon\colon K \to K$ and $\epsilon(K) \subset JK$. By Nakayama's lemma, the homomorphism $1 + \epsilon$ is an epimorphism. It follows that $1 + \epsilon$ is also a monomorphism [1, Chapter 8, p. 23] because $K$ is a noetherian module. So there exists $\beta'\colon K \to K$ such that $\beta'(1 + \epsilon) = \beta'\beta\alpha = 1$. We have obtained that the exact sequence $0 \to K \xrightarrow{\alpha} F \to M \to 0$ is splitting and $M$ is a projective module. The induction step will be omitted because of its routineness.

It suffices to prove the inequality $\mathrm{l\ gl\ dim}\ R \leq \sup \mathrm{inj\ dim}\ S$. If the right side of this formula is infinite there is nothing to do. Let $\sup \mathrm{inj\ dim}\ S = n < \infty$ and $M$ be an arbitrary finitely generated module. Then $\mathrm{Ext}^{n+1}(M, S) = 0$ for all modules $S$ which are annihilated by $J$. We obtain that $\mathrm{p\ dim}\ M \leq n$. So $\mathrm{l\ gl\ dim}\ R \leq n$ which was to be proved.

**Remark.** Small's result mentioned at the beginning of our paper easily follows from the above lemma. It suffices to use the formula

$$\mathrm{inj\ dim}_R S \leq \mathrm{r\ w\ dim}_R R/\mathfrak{A} + \mathrm{inj\ dim}_{R/\mathfrak{A}} S \qquad [2, \text{p. } 360].$$

**Theorem.** *Let $R$ be a left noetherian ring and $\mathfrak{A}$ a two-sided ideal contained in its Jacobson radical. If $\mathfrak{A}$ has the AR property, then*

$$\mathrm{l\ gl\ dim}\ R \leq p\ \mathrm{dim}\ R/\mathfrak{A} + \mathrm{l\ gl\ dim}\ R/\mathfrak{A}.$$

**Proof.** By Lemma 3, $\mathrm{l\ gl\ dim}\ R = \sup \mathrm{inj\ dim}\ S$, where supremum is taken over the category of modules which are annilated by $\mathfrak{A}$. We infer from Lemma 2 that $\mathrm{l\ gl\ dim}\ R \leq \sup \mathrm{p\ dim}\ S$, where supremum is taken over the above mentioned category. Our assertion will now follow from the formula

$$\mathrm{p\ dim}_R S \leq \mathrm{p\ dim}_R R/\mathfrak{A} + \mathrm{p\ dim}_{R/\mathfrak{A}} S \qquad [2, \text{p. } 360].$$

**Corollary.** *Let $R$ be a left noetherian semilocal ring and $J$ its Jacobson radical. If $J$ has the AR property, then* $\mathrm{l\ gl\ dim}\ R = \mathrm{p\ dim}\ R/J$.

In view of the above result one would like to mention the following open problem: Is it true that the Jacobson radical of every left and right noetherian ring $R$ has the left AR property? In this direction the author was able to prove that it is so in case $R$ is, moreover, semilocal and has the property that all its left ideals are two-sided.

I would like to express my thanks to the reviewer for his helpful remarks.

## REFERENCES

1. N. Bourbaki, *Éléments de mathématique*. XXIII. Part 1. *Les structures fondamentales de l'analyse*. Livre II: *Algèbre*. Chap. 8, Actualités Sci. Indust., no. 1261, Hermann, Paris, 1958. MR 20 #4576.

2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956. MR 17, 1040.

3. J. C. McConnell, *Localisation in enveloping rings*, J. London Math. Soc. 43 (1968), 421–428. MR 37 #4112.

4. K. L. Fields, *On the global dimension of residue rings*, Pacific J. Math. 32 (1970), 345–349. MR 42 #6049.

5. A. V. Jategaonkar, *Injective modules and localization in non-commutative noetherian rings*, Trans. Amer. Math. Soc. 190 (1974), 109–123.

6. Y. Nouazé and P. Gabriel, *Idéaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente*, J. Algebra 6 (1967), 77–99. MR 34 #5889.

7. L. W. Small, *A change of rings theorem*, Proc. Amer. Math. Soc. 19 (1968), 662–666. MR 36 #6460.

INSTITUTE OF MATHEMATICS, SNIADECKICH 8, WARSAW, POLAND