

A CHANGE OF RINGS THEOREM AND THE ARTIN-REES PROPERTY

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ABSTRACT. A two-sided ideal \mathfrak{A} of the ring R is said to have the left AR property if for every left ideal I and every k there exists an n such that $\mathfrak{A}^n \cap I \subset \mathfrak{A}^k I$. Let R be a left noetherian ring and \mathfrak{A} a two-sided ideal contained in its Jacobson radical. If \mathfrak{A} has the AR property then $l\text{ gl dim } R \leq p\text{ dim } R/\mathfrak{A} + l\text{ gl dim } R/\mathfrak{A}$, where $p\text{ dim } R/\mathfrak{A}$ denotes the (left) projective dimension of the module R/\mathfrak{A} .

Let R be a left noetherian ring and J its Jacobson radical. Small has proved in [7] that $l\text{ gl dim } R \leq r\text{ w dim } R/\mathfrak{A} + l\text{ gl dim } R/\mathfrak{A}$, where \mathfrak{A} is a twosided ideal contained in J . The natural question arises whether in the above formula right weak dimension can be replaced by left projective dimension. Fields [4] has produced an example which shows that the answer, in general, is "no". However we are able to prove that in case \mathfrak{A} satisfies the AR property the answer is "yes".

Our result is derived from three lemmas. The first is a homological characterization of the AR property which may be of some independent interest and the third is an improvement of Small's result.

Definition. A two-sided ideal \mathfrak{A} of the ring R is said to have the (left) AR property if for every left ideal I and every k there exists an n such that $\mathfrak{A}^n \cap I \subset \mathfrak{A}^k I$.

Remark. In the commutative noetherian case it follows from the Artin-Rees lemma that every ideal has the AR property.

In the sequel, $E(M)$ will denote an injective hull of the left module M .

Lemma 1. *Let R be a left noetherian ring and \mathfrak{A} its arbitrary two-sided ideal. Then the following two conditions are equivalent.*

- 1°. \mathfrak{A} has the AR property.
- 2°. $E(M) = \bigcup_i (0: \mathfrak{A}^i)$ for any left module M whose every element is annihilated by some power of \mathfrak{A} where $(0: \mathfrak{A}^i) = \{x \in E(M) \mid \mathfrak{A}^i x = 0\}$.

Proof. $1^\circ \rightarrow 2^\circ$. We put $E_1 = \bigcup_i (0: \mathfrak{A}^i)$. It will be sufficient to prove that E_1 is injective because the latter contains M . Let $f: I \rightarrow E_1$ where

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I is an arbitrary left ideal of the ring R . $f(I) \subset (0: \mathfrak{A}^k)$ for some k because I is finitely generated. We pick n such that $\mathfrak{A}^n \cap I \subset \mathfrak{A}^k I$. $\mathfrak{A}^n \cap I$ is contained in the kernel of f because $f(\mathfrak{A}^n \cap I) \subset f(\mathfrak{A}^k I) = \mathfrak{A}^k f(I) = 0$.

We have the diagram of R/\mathfrak{A}^n -modules with an exact row:

$$\begin{array}{ccccc} 0 & \longrightarrow & I/\mathfrak{A}^n \cap I & \longrightarrow & R/\mathfrak{A}^n \\ & & \searrow \bar{f} & & \\ & & & & (0: \mathfrak{A}^n) \end{array}$$

\bar{f} is the homomorphism induced by f (we can assume that $n > k$). $(0: \mathfrak{A}^n)$ is an injective R/\mathfrak{A}^n -module. So there exists a homomorphism $R/\mathfrak{A}^n \rightarrow (0: \mathfrak{A}^n)$ which makes the above diagram commutative. This homomorphism composed with the natural homomorphism $R \rightarrow R/\mathfrak{A}^n$ is the desired extension of f to R .

$2^\circ \rightarrow 1^\circ$. Let us consider the natural homomorphism $f: I \rightarrow I/\mathfrak{A}^k I \subset E(I/\mathfrak{A}^k I)$. There exists an element $x \in E(I/\mathfrak{A}^k I)$ such that $f(r) = rx$ for all $r \in I$. By our assumption $x \in (0: \mathfrak{A}^n)$ for some n . It follows that $\mathfrak{A}^n \cap I$ is contained in the kernel of f , which coincides with $\mathfrak{A}^k I$, which was to be proved. Several cases of the above lemma are known [3], [5], [6].

In the sequel $\text{inj dim } M$, $\text{p dim } M$ will denote the injective and the projective dimension of the left module M respectively.

Lemma 2. *Let R be a left noetherian ring and \mathfrak{A} its two-sided ideal which has the AR property. If M is a module whose every element is annihilated by some power of \mathfrak{A} then $\text{inj dim } M \leq \sup \text{p dim } S$, where supremum is taken over the category of modules which are annihilated by \mathfrak{A} .*

Proof. It will suffice to prove that if $\text{Ext}^{n+1}(S, M) = 0$ for all modules S which are annihilated by \mathfrak{A} , then $\text{inj dim } M \leq n$. We start with $n = 0$. If $E(M)/M \neq 0$, then, by Lemma 1, this module contains a nonzero submodule M_1/M which is annihilated by \mathfrak{A} . We have the exact sequence $0 \rightarrow M \rightarrow M_1 \rightarrow M_1/M \rightarrow 0$ which is nonsplitting because $E(M)$ is an essential extension of M (i.e. every nonzero submodule of $E(M)$ has a nonzero intersection with M). This exact sequence represents a nonzero element of $\text{Ext}^1(M_1/M, M)$ which contradicts our hypothesis.

Let $n > 0$. We consider the exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$, $\text{Ext}^n(S, E(M)/M) = \text{Ext}^{n+1}(S, M) = 0$. By Lemma 1, each element of $E(M)/M$ is annihilated by some power of \mathfrak{A} . So by the inductive hypothesis, $\text{inj dim } E(M)/M \leq n - 1$. We infer that $\text{inj dim } M \leq n$, which was to be proved.

Lemma 3. *Let R be a left noetherian ring and J its Jacobson radical. Then $1 \text{ gl dim } R = \sup \text{inj dim } S$, where supremum is taken over the category of modules which are annihilated by J .*

Proof. Let M be a finitely generated module. We shall prove that the condition $\text{Ext}^{n+1}(M, S) = 0$ for all modules S which are annihilated by J

implies that $\text{p dim } M \leq n$. Let $n = 0$. We have an exact sequence $0 \rightarrow K \xrightarrow{\alpha} F \rightarrow M \rightarrow 0$, where F is a finitely generated free module. The map induced by α , $\text{Hom}(F, K/JK) \rightarrow \text{Hom}(K, K/JK)$, is an epimorphism because $\text{Ext}^1(M, K/JK) = 0$. In particular the natural homomorphism $K \rightarrow K/JK$ which will be denoted by γ can be extended to the homomorphism $\delta: F \rightarrow K/JK$. F is a free module, so there exist $\beta: F \rightarrow K$ such that $\gamma\beta = \delta$. We obtain that $\gamma\beta\alpha = \gamma$. Let us put $\epsilon = \beta\alpha - 1$. Then $\epsilon: K \rightarrow K$ and $\epsilon(K) \subset JK$. By Nakayama's lemma, the homomorphism $1 + \epsilon$ is an epimorphism. It follows that $1 + \epsilon$ is also a monomorphism [1, Chapter 8, p. 23] because K is a noetherian module. So there exists $\beta': K \rightarrow K$ such that $\beta'(1 + \epsilon) = \beta'\beta\alpha = 1$. We have obtained that the exact sequence $0 \rightarrow K \xrightarrow{\alpha} F \rightarrow M \rightarrow 0$ is splitting and M is a projective module. The induction step will be omitted because of its routineness.

It suffices to prove the inequality $1 \text{ gl dim } R \leq \sup \text{inj dim } S$. If the right side of this formula is infinite there is nothing to do. Let $\sup \text{inj dim } S = n < \infty$ and M be an arbitrary finitely generated module. Then $\text{Ext}^{n+1}(M, S) = 0$ for all modules S which are annihilated by J . We obtain that $\text{p dim } M \leq n$. So $1 \text{ gl dim } R \leq n$ which was to be proved.

Remark. Small's result mentioned at the beginning of our paper easily follows from the above lemma. It suffices to use the formula

$$\text{inj dim}_R S \leq \text{r w dim}_R R/\mathfrak{A} + \text{inj dim}_{R/\mathfrak{A}} S \quad [2, \text{p. 360}].$$

Theorem. *Let R be a left noetherian ring and \mathfrak{A} a two-sided ideal contained in its Jacobson radical. If \mathfrak{A} has the AR property, then*

$$1 \text{ gl dim } R \leq \text{p dim } R/\mathfrak{A} + 1 \text{ gl dim } R/\mathfrak{A}.$$

Proof. By Lemma 3, $1 \text{ gl dim } R = \sup \text{inj dim } S$, where supremum is taken over the category of modules which are annihilated by \mathfrak{A} . We infer from Lemma 2 that $1 \text{ gl dim } R \leq \sup \text{p dim } S$, where supremum is taken over the above mentioned category. Our assertion will now follow from the formula

$$\text{p dim}_R S \leq \text{p dim}_R R/\mathfrak{A} + \text{p dim}_{R/\mathfrak{A}} S \quad [2, \text{p. 360}].$$

Corollary. *Let R be a left noetherian semilocal ring and J its Jacobson radical. If J has the AR property, then $1 \text{ gl dim } R = \text{p dim } R/J$.*

In view of the above result one would like to mention the following open problem: Is it true that the Jacobson radical of every left and right noetherian ring R has the left AR property? In this direction the author was able to prove that it is so in case R is, moreover, semilocal and has the property that all its left ideals are two-sided.

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REFERENCES

1. N. Bourbaki, *Éléments de mathématique*. XXIII. Part 1. *Les structures fondamentales de l'analyse*. Livre II: *Algèbre*. Chap. 8, Actualités Sci. Indust., no. 1261, Hermann, Paris, 1958. MR 20 #4576.
2. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956. MR 17, 1040.
3. J. C. McConnell, *Localisation in enveloping rings*, J. London Math. Soc. 43 (1968), 421–428. MR 37 #4112.
4. K. L. Fields, *On the global dimension of residue rings*, Pacific J. Math. 32 (1970), 345–349. MR 42 #6049.
5. A. V. Jategaonkar, *Injective modules and localization in non-commutative noetherian rings*, Trans. Amer. Math. Soc. 190 (1974), 109–123.
6. Y. Nouazé and P. Gabriel, *Idéaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente*, J. Algebra 6 (1967), 77–99. MR 34 #5889.
7. L. W. Small, *A change of rings theorem*, Proc. Amer. Math. Soc. 19 (1968), 662–666. MR 36 #6460.

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