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A LATTICE THEORETIC CHARACTERIZATION OF AN INTEGRAL OPERATOR[1]

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ABSTRACT. We are concerned here with obtaining necessary and sufficient conditions for a linear operator, $K: \mathfrak{L}(X_1, \mathfrak{A}_1, \mu_1) \to M(X_2, \mathfrak{A}_2, \mu_2)$, to be represented by an integral, $K(f) = \int k(x, y) f(y)\, dy$, with an $\mathfrak{A}_2 \times \mathfrak{A}_1$ measurable kernel $k(x, y)$. That such conditions are developed in a lattice theoretic context will be shown to be quite natural. Our direction will be to characterize an integral operator by its action pointwise: i.e., $K(\cdot)(x)$ is a linear functional on a subspace of the essentially bounded functions. Such a development leads one to define the kernel, $k(x, y)$, in a pointwise fashion also, and as a result we are confronted with the question of the $\mathfrak{A}_2 \times \mathfrak{A}_1$ measurability of $k(x, y)$.

Definitions and notation. The following definitions, unless otherwise noted, may be found in [1] and [2].

Definition. The real vector space R is called an ordered vector space when R is partially ordered by $\leq$ and satisfies for $x, y, z \in R$,

(1) $x \leq y$ implies $x + z \leq y + z$,

(2) $x \geq 0$ implies $rx \geq 0$ for any real number $r \geq 0$.

The ordered vector space R is called a Riesz space when for each $x, y \in R$ the least upper bound of x and y, written $x \vee y$, exists in R. The Riesz space $(R, \leq)$ is called Dedekind complete if for any subset $\{x_\alpha\} \subset R$ such that there is an upper bound $y \in R$ for $\{x_\alpha\}$, then the least upper bound of $\{x_\alpha\}$ exists in R: written $\sup\{x_\alpha\} \in R$. A sequence $\langle x_n \rangle$ on R is said to converge in order to $x \in R$, written $x_n \to x(0)$, whenever $\underline{\lim}\, x_n = \overline{\lim}\, x_n = x$ where

$$\underline{\lim}\, x_n = \sup_n \inf_{k \geq n} x_k, \quad \overline{\lim}\, x_n = \inf_n \sup_{k \geq n} x_k,$$

and inf as usual means greatest lower bound. By $0 \leq x_n \uparrow x$ we mean $0 \leq x_n \leq x_{n+1}$ and $\sup_n x_n = x$; $x_n \downarrow x$ means $x_n \geq x_{n+1}$ and $\inf_n x_n = x$. A linear mapping $T: R_1 \to R_2$ between Riesz spaces is called (0)-continuous when it maps order convergent sequences into order convergent sequences, and T is called positive when $x \geq 0$ implies $T(x) \geq 0$.

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Let $(\mathbf{X}, \mathfrak{A}, \mu)$ be a σ-finite measure space and denote by $M(\mathbf{X})$ the collection of all equivalence classes of $\mathfrak{A}$ measurable finite μ a.e. real valued functions on $\mathbf{X}$ modulo μ null functions. For $f, g \in M(\mathbf{X})$ define $f \leq g$ providing $f(x) \leq g(x)$ for μ a.e. $x \in \mathbf{X}$, then $(M(\mathbf{X}), \leq)$ is a Dedekind complete Riesz space: see [1, p. 126] or [3, p. 335]. A linear subspace $\mathfrak{L} \subset M(\mathbf{X})$ is called an ideal when $g \in M(\mathbf{X})$, $f \in \mathfrak{L}$ and $|g| \leq |f|$ implies $g \in \mathfrak{L}$. We say that a sequence $\pi = \langle x_j \rangle$ is admissible for $\mathfrak{L}$ when $\{x_j\}$ is a countable collection of disjoint, measurable sets of finite measure where $\bigcup x_j = \mathbf{X}$ and $\chi(x_j) \in \mathfrak{L}$. For a set E, $\chi(E)$ is the characteristic function of E. We shall be required to distinguish between equivalence classes of functions and functions which are defined and finite everywhere. Let ψ be the canonical homomorphism that sends a function f, defined and finite real valued a.e., to its equivalence class $\langle f \rangle$: $\psi(f) = \langle f \rangle$. For $S \subset M(\mathbf{X})$, let $\overline{S} = \{f | f: \mathbf{X} \to (-\infty, \infty)$ and $\psi(f) \in S\}$. We may partially order $\overline{M(\mathbf{X})}$ as follows: for $f, g \in \overline{M(\mathbf{X})}$, $f \leq g$ if and only if for all $x \in \mathbf{X}$, $f(x) \leq g(x)$. If S is an ideal of $M(\mathbf{X})$, then $\overline{S}$ is an ideal of $\overline{M(\mathbf{X})}$. Although $M(\mathbf{X})$ is Dedekind complete, in general $\overline{M(\mathbf{X})}$ is not Dedekind complete. For L an ideal of $M(\mathbf{X})$, f_n, $f \in \overline{L}$, we have $f_n \to f(0)$ in $\overline{L}$ when there exists $g \in \overline{L}$ such that for all n and all $x \in X$, $|f_n(x)| \leq g(x)$ and $\lim f_n(x) = f(x)$ [2, p. 64].

For the duration of this paper we shall assume that $(\mathbf{X}_1, \mathfrak{A}_1, \mu_1)$ and $(\mathbf{X}_2, \mathfrak{A}_2, \mu_2)$ are σ-finite measure spaces with respect to nonnegative, countably additive, extended real valued set functions μ_1 and μ_2, respectively. We also assume that $(\mathbf{X}_1, \mathfrak{A}_1, \mu_1)$ is a separable measure space. We shall denote by $\mathfrak{L}$ an ideal of $M(\mathbf{X}_1)$ with an admissible sequence $\pi = \langle X_j \rangle$.

Let $T: \mathfrak{L} \to M(\mathbf{X}_2)$, S a linear subspace of $\mathfrak{L}$ and $\hat{T}: S \to \overline{M(\mathbf{X}_2)}$; then $\hat{T}$ is called a lift of T on S when for all $f \in S$, $\psi \circ \hat{T}(f) = T(f)$. If a lift $\hat{T}$ of T on S exists, it need not be unique, also $\hat{T}$ need not inherit even the simplest properties of T: linearity, positivity, order continuity.

A map $K: \mathfrak{L} \to M(\mathbf{X}_2)$ is called an integral operator when there is $k(x, y) \in M(\mathbf{X}_2 \times \mathbf{X}_1)$ such that for $f \in \mathfrak{L}$,

$$K(f)(x) = \int k(x, y) f(y)\, d\mu_1(y).$$

k is called the kernel of K, and the association is denoted $K = [k]$. That an ideal $\mathfrak{L}$ of $M(\mathbf{X}_1)$ with an admissible sequence π is a natural domain for an integral operator may be inferred from [4] and [7]. For each $f \in \mathfrak{L}$ and a.e. $x \in X_2$ fixed, $k(x, y)f(y)$ is integrable on X_1; so for a.e. x,

$$\int |k(x, y)| f(y)\, d\mu_1(y) < \infty$$

and, by Fubini's theorem, defines a function in $M(\mathbf{X}_2)$. Thus $K: \mathfrak{L} \to M(\mathbf{X}_2)$ can be considered the difference of two positive operators $[k^+]$, $[k^-]$: i.e.,

$$K(f) = \int k^+(x, y) f(y)\, dy - \int k^-(x, y) f(y)\, dy$$

where $[k^{\pm}]: \mathfrak{L} \to M(\mathbf{X}_2)$ are positive operators.

The following theorem gives necessary and sufficient conditions for an operator $T: \mathfrak{L} \to M(\mathbf{X}_2)$ to be an integral operator. Since an integral operator is necessarily the difference of two positive operators, it will suffice to consider positive operators only.

Theorem. *Let $T: \mathfrak{L} \to M(\mathbf{X}_2)$ be a positive linear operator; then there exists $k \in M(\mathbf{X}_2 \times \mathbf{X}_1)$ such that T is an integral operator, $T = [k]$ and $0 \leq k$ if and only if*

(1) *$T: \mathfrak{L} \to M(\mathbf{X}_2)$ is (0)-continuous,*

(2) *for each j there is a lift $\hat{T}_j$ of T on $L_\infty(x_j)$ that is positive, linear and order continuous.*

Proof. Let $T = [k]$ be a positive linear integral operator with kernel $0 \leq k$. By [2, p. 215], it is sufficient to show that if $f_n \in \mathfrak{L}$ and $0 \leq f_n \downarrow 0$, then $0 \leq T(f_n) \downarrow 0$, to obtain order continuity for T. This follows easily from the Lebesgue dominated convergence theorem. To verify (2) let k_0 be a specific kernel: $k_0 \in k \cap \overline{M(\mathbf{X}_2 \times \mathbf{X}_1)}$ such that $0 \leq k_0(x, y)$ for every $(x, y) \in \mathbf{X}_2 \times \mathbf{X}_1$. For each j, $\int k_0(x, y)\chi(x_j)(y)\,dy < \infty$ except for $x \in A_j$ where $\mu_2(A_j) = 0$. Now take $k_1(x, y) = k_0(x, y)$ for $(x, y) \in \widetilde{\bigcup A_j} \times \mathbf{X}_1$ and define $k_1(x, y) = 0$ for $(x, y) \in \bigcup A_j \times \mathbf{X}_1$. Thus k_1 and k_0 differ on a set of $\mu_2 \times \mu_1$ measure zero. If $f \in L_\infty(x_j)$, then for some $0 < c$ we have $|f| \leq c\chi(x_j)$. Consequently,

$$\left|\int k_1(x, y) f(y)\,dy\right| \leq \int k_1(x, y) \cdot c\chi(x_j)(y)\,dy < \infty$$

for all $x \in \mathbf{X}_2$. For $f \in L_\infty(x_j)$, define

$$\hat{T}_j(f) = \int k_1(x, y) f(y)\,dy;$$

then $\hat{T}_j$ is a lift of T on $L_\infty(x_j)$. It is obvious that $\hat{T}_j$ is positive and linear. If $f_n \in L_\infty(x_j)$, $0 \leq f_n \downarrow 0$, then a simple application of Lebesgue's dominated convergence theorem shows that $0 \leq \hat{T}_j(f_n)(x) \downarrow 0$ for all x and $\hat{T}_j$ is order continuous.

Now let us suppose that $T: \mathfrak{L} \to M(\mathbf{X}_2)$ is a positive, linear and order continuous map that satisfies (2). We shall construct the integral representation for T by viewing "$\hat{T}_j(\ )(x)$" as a measure on the relativized space $(X_j, \mathfrak{A}_1 \cap X_j, \mu_1)$ and applying the Radon-Nikodym theorem. Let $\mathfrak{A}_j = \{E \cap X_j : E \in \mathfrak{A}_1\}$ and $x \in \mathbf{X}_2$ be fixed, then define for $E \in \mathfrak{A}_j$, $\mu_{x_j}(E) = \hat{T}_j(\langle\chi(E)\rangle)(x)$. The finite additivity of μ_{x_j} follows from the linearity of $\hat{T}_j$, and the nonnegativity of μ_{x_j} comes from the positivity of $\hat{T}_j$. Let $\langle E_i\rangle$ be a countable disjoint sequence from $\mathfrak{A}_j$; then $\chi(\bigcup_{i=1}^\infty E_i) - \chi(\bigcup_{i=1}^n E_i) = f_n$ and $0 \leq f_n \downarrow 0$. Since $\hat{T}_j$ is order continuous, $0 \leq \hat{T}_j(f_n)(x) \downarrow 0$ for all $x \in \mathbf{X}_2$. Consequently, $\lim_{n\to\infty} \mu_{x_j}(\bigcup_1^\infty E_i) - \Sigma_{i=1}^n \mu_{x_j}(E_i) = 0$: μ_{x_j} is countably additive.

Now if $E \in \mathfrak{A}_j$ is a μ_1 null set, then

$$\hat{T}_j(\langle\chi(E)\rangle)(x) = \hat{T}(2\langle\chi(E)\rangle)(x) = 2\hat{T}(\langle\chi(E)\rangle)(x)$$

for all $x \in \mathbf{X}_2$, so $\hat{T}_j(\langle\chi(E)\rangle)(x) = 0$ for all $x \in \mathbf{X}_2$ and μ_{x_j} is absolutely continuous with respect to μ_1. Let $k_j(x, y)$ be the Radon-Nikodym derivative of μ_{x_j} with respect to μ_1. Clearly $0 \leq k_j(x, y)$ for all x and y. Since $(\mathbf{X}_1, \mathfrak{A}_1, \mu_1)$ is separable, it follows by [5, p. 616], that $k_j(x, y)$ is $\mathfrak{A}_2 \times \mathfrak{A}_1$ measurable.

For $E \in \mathfrak{A}_j$ we have $\mu_{x_j}(E) = \int k_j(x, y)\chi(E)(y)\, dy$. Thus for any simple function $r = \sum_{i=1}^n c_i \chi(E_i)$ where $\{E_i\}_{i=1}^n$ are disjoint and $E_i \subset X_j$,

$$\hat{T}_j(\langle r\rangle)(x) = \sum_{i=1}^n c_i T_j(\langle\chi(E_i)\rangle)(x) = \sum_{i=1}^n c_i \mu_{x_j}(E_i) = \int k_j(x, y) r(y)\, dy$$

for all $x \in \mathbf{X}_2$. Thus

$$T(r) = \psi \circ \int k_j(x, y) r(y)\, dy$$

for any simple function r vanishing outside X_j.

Now let $0 \leq f \in \mathfrak{L}$ and f vanish off of X_j; then there exists a sequence $\langle f_n \rangle$ of simple functions such that $0 \leq f_n \uparrow f$ and f_n vanishes off of X_j. So

$$T(f) = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} \psi \circ \int k_j(x, y) f_n(y)\, dy = \psi \circ \int k_j(x, y) f(y)\, dy.$$

If we drop the distinction between equivalence classes and functions, we now have for all $f \in \mathfrak{L}$ that vanish off of X_j that $T(f) = \int k_j(x, y) f(y)\, dy$. The extension from nonnegative f to general f, as usual, uses the decomposition $f = f^+ - f^-$ where $f^+ = f \vee 0$ and $f^- = -f \vee 0$.

If we set $k = \sum_{j=1}^\infty k_j$, then $0 \leq k$, and k is $\mathfrak{A}_2 \times \mathfrak{A}_1$ measurable. Since $k\chi(x_j \times X_1) = k_j$, $k < \infty$, $\mu_2 \times \mu_1$ a.e. Now let $0 \leq f \in \mathfrak{L}$ and $f = \sum_{j=1}^\infty f_j$ where $0 \leq f_j = f \cdot \chi(x_j)$. Since $0 \leq \sum_{j=1}^n f_j \uparrow f$,

$$\begin{aligned} T(f) &= \sum_{j=1}^\infty T(f_j) = \sum_{j=1}^\infty \int k_j(x, y) f(y)\, dy \\ &= \sum_{j=1}^\infty \int k(x, y) f(y) \chi(x_j)(y)\, dy = \int k(x, y) f(y)\, dy. \end{aligned}$$

The extension from nonnegative f to general $f \in \mathfrak{L}$, as before, uses $f = f^+ - f^-$.

In this paragraph we provide an example of a lift of a rather well-known operator. Let $(\mathbf{X}, \mathfrak{A}, \mu)$ be the usual Lebesgue measure on the real line $\mathbf{X}$, and let L_2 be the square integrable functions defined on $\mathbf{X}$. Clearly L_2 is an ideal of $M(\mathbf{X})$ having an admissible sequence. An integral operator K: $L_2 \to L_2$, where $K = [k]$ is of Hilbert-Schmidt class when $\iint |k(x, y)|^2\, dx\, dy < \infty$: see [8]. Let us suppose $0 \leq k$, and choose any specific representative

k_1 of k. So k_1 is square integrable and there is a μ null set A such that if $x \notin A$, then $0 \leq \int k_1(x, y)f(y)\,dy < \infty$. Now define $k_0(x, y) = k_1(x, y)$ when $x \notin A$ and $k_0(x, y) = 0$ when $x \in A$. Thus $k_1 - k_0$ is $\mu \times \mu$ null function and $K = [\langle k_0 \rangle]$. We may now define a lift $\hat{K}$ of K on L_2 by $\hat{K}(f)(x) = \int k_0(x, y)f(y)\,dy$: i.e. for all $f \in L_2$ and for all $x \in \mathbf{X}$, $0 \leq \int k_0(x, y)f(y)\,dy < \infty$.

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