

A LATTICE THEORETIC CHARACTERIZATION OF AN INTEGRAL OPERATOR¹

LAWRENCE LESSNER

ABSTRACT. We are concerned here with obtaining necessary and sufficient conditions for a linear operator, $K: \mathfrak{L}(X_1, \mathfrak{A}_1, \mu_1) \rightarrow M(X_2, \mathfrak{A}_2, \mu_2)$, to be represented by an integral, $K(f) = \int k(x, y)f(y) dy$, with an $\mathfrak{A}_2 \times \mathfrak{A}_1$ measurable kernel $k(x, y)$. That such conditions are developed in a lattice theoretic context will be shown to be quite natural. Our direction will be to characterize an integral operator by its action pointwise: i.e., $K(\cdot \chi_x)$ is a linear functional on a subspace of the essentially bounded functions. Such a development leads one to define the kernel, $k(x, y)$, in a pointwise fashion also, and as a result we are confronted with the question of the $\mathfrak{A}_2 \times \mathfrak{A}_1$ measurability of $k(x, y)$.

Definitions and notation. The following definitions, unless otherwise noted, may be found in [1] and [2].

Definition. The real vector space R is called an ordered vector space when R is partially ordered by \leq and satisfies for $x, y, z \in R$,

- (1) $x \leq y$ implies $x + z \leq y + z$,
- (2) $x \geq 0$ implies $rx \geq 0$ for any real number $r \geq 0$.

The ordered vector space R is called a Riesz space when for each $x, y \in R$ the least upper bound of x and y , written $x \vee y$, exists in R . The Riesz space (R, \leq) is called Dedekind complete if for any subset $\{x_\alpha\} \subset R$ such that there is an upper bound $y \in R$ for $\{x_\alpha\}$, then the least upper bound of $\{x_\alpha\}$ exists in R : written $\sup\{x_\alpha\} \in R$. A sequence $\{x_n\}$ on R is said to converge in order to $x \in R$, written $x_n \rightarrow x(0)$, whenever $\underline{\lim} x_n = \overline{\lim} x_n = x$ where

$$\underline{\lim} x_n = \sup_n \inf_{k \geq n} x_k, \quad \overline{\lim} x_n = \inf_n \sup_{k \geq n} x_k,$$

and \inf as usual means greatest lower bound. By $0 \leq x_n \uparrow x$ we mean $0 \leq x_n \leq x_{n+1}$ and $\sup_n x_n = x$; $x_n \downarrow x$ means $x_n \geq x_{n+1}$ and $\inf_n x_n = x$. A linear mapping $T: R_1 \rightarrow R_2$ between Riesz spaces is called (0)-continuous when it maps order convergent sequences into order convergent sequences, and T is called positive when $x \geq 0$ implies $T(x) \geq 0$.

Received by the editors November 10, 1972 and, in revised form, March 12, 1973.
 AMS (MOS) subject classifications (1970). Primary 47A65, 47B55, 47C99, 47O15, 46G15.

Key words and phrases. Integral operator, lift, Riesz space.

¹ This work is part of a doctoral thesis written under the direction of Dr. Alan Schumitzky at the University of Southern California.

Let (X, \mathcal{A}, μ) be a σ -finite measure space and denote by $M(X)$ the collection of all equivalence classes of \mathcal{A} measurable finite μ a.e. real valued functions on X modulo μ null functions. For $f, g \in M(X)$ define $f \leq g$ providing $f(x) \leq g(x)$ for μ a.e. $x \in X$, then $(M(X), \leq)$ is a Dedekind complete Riesz space: see [1, p. 126] or [3, p. 335]. A linear subspace $\mathcal{L} \subset M(X)$ is called an ideal when $g \in M(X), f \in \mathcal{L}$ and $|g| \leq |f|$ implies $g \in \mathcal{L}$. We say that a sequence $\pi = \langle x_j \rangle$ is admissible for \mathcal{L} when $\{x_j\}$ is a countable collection of disjoint, measurable sets of finite measure where $\bigcup x_j = X$ and $\chi(x_j) \in \mathcal{L}$. For a set $E, \chi(E)$ is the characteristic function of E . We shall be required to distinguish between equivalence classes of functions and functions which are defined and finite everywhere. Let ψ be the canonical homomorphism that sends a function f , defined and finite real valued a.e., to its equivalence class $\langle f \rangle: \psi(f) = \langle f \rangle$. For $S \subset M(X)$, let $\bar{S} = \{f|f: X \rightarrow (-\infty, \infty) \text{ and } \psi(f) \in S\}$. We may partially order $\overline{M(X)}$ as follows: for $f, g \in \overline{M(X)}, f \leq g$ if and only if for all $x \in X, f(x) \leq g(x)$. If S is an ideal of $M(X)$, then \bar{S} is an ideal of $\overline{M(X)}$. Although $M(X)$ is Dedekind complete, in general $\overline{M(X)}$ is not Dedekind complete. For L an ideal of $M(X), f_n, f \in \bar{L}$, we have $f_n \rightarrow f(0)$ in \bar{L} when there exists $g \in \bar{L}$ such that for all n and all $x \in X, |f_n(x)| \leq g(x)$ and $\lim f_n(x) = f(x)$ [2, p. 64].

For the duration of this paper we shall assume that $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ are σ -finite measure spaces with respect to nonnegative, countably additive, extended real valued set functions μ_1 and μ_2 , respectively. We also assume that $(X_1, \mathcal{A}_1, \mu_1)$ is a separable measure space. We shall denote by \mathcal{L} an ideal of $M(X_1)$ with an admissible sequence $\pi = \langle X_j \rangle$.

Let $T: \mathcal{L} \rightarrow M(X_2), S$ a linear subspace of \mathcal{L} and $\hat{T}: S \rightarrow \overline{M(X_2)}$; then \hat{T} is called a lift of T on S when for all $f \in S, \psi \circ \hat{T}(f) = T(f)$. If a lift \hat{T} of T on S exists, it need not be unique, also \hat{T} need not inherit even the simplest properties of T : linearity, positivity, order continuity.

A map $K: \mathcal{L} \rightarrow M(X_2)$ is called an integral operator when there is $k(x, y) \in M(X_2 \times X_1)$ such that for $f \in \mathcal{L}$,

$$K(f)(x) = \int k(x, y)f(y) d\mu_1(y).$$

k is called the kernel of K , and the association is denoted $K = [k]$. That an ideal \mathcal{L} of $M(X_1)$ with an admissible sequence π is a natural domain for an integral operator may be inferred from [4] and [7]. For each $f \in \mathcal{L}$ and a.e. $x \in X_2$ fixed, $k(x, y)f(y)$ is integrable on X_1 ; so for a.e. x ,

$$\int |k(x, y)|f(y) d\mu_1(y) < \infty$$

and, by Fubini's theorem, defines a function in $M(X_2)$. Thus $K: \mathcal{L} \rightarrow M(X_2)$ can be considered the difference of two positive operators $[k^+], [k^-]$: i.e.,

$$K(f) = \int k^+(x, y)f(y) d\mu_1(y) - \int k^-(x, y)f(y) d\mu_1(y)$$

where $[k^\pm]: \mathcal{L} \rightarrow M(X_2)$ are positive operators.

The following theorem gives necessary and sufficient conditions for an operator $T: \mathcal{L} \rightarrow M(X_2)$ to be an integral operator. Since an integral operator is necessarily the difference of two positive operators, it will suffice to consider positive operators only.

Theorem. *Let $T: \mathcal{L} \rightarrow M(X_2)$ be a positive linear operator; then there exists $k \in M(X_2 \times X_1)$ such that T is an integral operator, $T = [k]$ and $0 \leq k$ if and only if*

- (1) $T: \mathcal{L} \rightarrow M(X_2)$ is (0)-continuous,
- (2) for each j there is a lift \hat{T}_j of T on $L_\infty(x_j)$ that is positive, linear and order continuous.

Proof. Let $T = [k]$ be a positive linear integral operator with kernel $0 \leq k$. By [2, p. 215], it is sufficient to show that if $f_n \in \mathcal{L}$ and $0 \leq f_n \downarrow 0$, then $0 \leq T(f_n) \downarrow 0$, to obtain order continuity for T . This follows easily from the Lebesgue dominated convergence theorem. To verify (2) let k_0 be a specific kernel: $k_0 \in k \cap \overline{M(X_2 \times X_1)}$ such that $0 \leq k_0(x, y)$ for every $(x, y) \in X_2 \times X_1$. For each j , $\int k_0(x, y)\chi(x_j)(y) dy < \infty$ except for $x \in A_j$ where $\mu_2(A_j) = 0$. Now take $k_1(x, y) = k_0(x, y)$ for $(x, y) \in \widetilde{\bigcup} A_j \times X_1$ and define $k_1(x, y) = 0$ for $(x, y) \in \bigcup A_j \times X_1$. Thus k_1 and k_0 differ on a set of $\mu_2 \times \mu_1$ measure zero. If $f \in L_\infty(x_j)$, then for some $0 < c$ we have $|f| \leq c\chi(x_j)$. Consequently,

$$\left| \int k_1(x, y)f(y) dy \right| \leq \int k_1(x, y) \cdot c\chi(x_j)(y) dy < \infty$$

for all $x \in X_2$. For $f \in L_\infty(x_j)$, define

$$\hat{T}_j(f) = \int k_1(x, y)f(y) dy;$$

then \hat{T}_j is a lift of T on $L_\infty(x_j)$. It is obvious that \hat{T}_j is positive and linear. If $f_n \in L_\infty(x_j)$, $0 \leq f_n \downarrow 0$, then a simple application of Lebesgue's dominated convergence theorem shows that $0 \leq \hat{T}_j(f_n)(x) \downarrow 0$ for all x and \hat{T}_j is order continuous.

Now let us suppose that $T: \mathcal{L} \rightarrow M(X_2)$ is a positive, linear and order continuous map that satisfies (2). We shall construct the integral representation for T by viewing " $\hat{T}_j(\cdot)(x)$ " as a measure on the relativized space $(X_j, \mathcal{A}_1 \cap X_j, \mu_1)$ and applying the Radon-Nikodym theorem. Let $\mathcal{A}_j = \{E \cap X_j : E \in \mathcal{A}_1\}$ and $x \in X_2$ be fixed, then define for $E \in \mathcal{A}_j$, $\mu_{x_j}(E) = \hat{T}_j(\chi(E))(x)$. The finite additivity of μ_{x_j} follows from the linearity of \hat{T}_j , and the nonnegativity of μ_{x_j} comes from the positivity of \hat{T}_j . Let $\langle E_i \rangle$ be a countable disjoint sequence from \mathcal{A}_j ; then $\chi(\bigcup_{i=1}^\infty E_i) - \chi(\bigcup_{i=1}^n E_i) = f_n$ and $0 \leq f_n \downarrow 0$. Since \hat{T}_j is order continuous, $0 \leq \hat{T}_j(f_n)(x) \downarrow 0$ for all $x \in X_2$.

Consequently, $\lim_{n \rightarrow \infty} \mu_{x_j}(\bigcup_{i=1}^\infty E_i) - \sum_{i=1}^n \mu_{x_j}(E_i) = 0$; μ_{x_j} is countably additive.

Now if $E \in \mathfrak{A}_j$ is a μ_1 null set, then

$$\hat{T}_j(\langle \chi(E) \rangle)(x) = \hat{T}(2\langle \chi(E) \rangle)(x) = 2\hat{T}(\langle \chi(E) \rangle)(x)$$

for all $x \in X_2$, so $\hat{T}_j(\langle \chi(E) \rangle)(x) = 0$ for all $x \in X_2$ and μ_{x_j} is absolutely continuous with respect to μ_1 . Let $k_j(x, y)$ be the Radon-Nikodym derivative of μ_{x_j} with respect to μ_1 . Clearly $0 \leq k_j(x, y)$ for all x and y . Since $(X_1, \mathfrak{A}_1, \mu_1)$ is separable, it follows by [5, p. 616], that $k_j(x, y)$ is $(\mathfrak{A}_2 \times \mathfrak{A}_1)$ measurable.

For $E \in \mathfrak{A}_j$ we have $\mu_{x_j}(E) = \int k_j(x, y)\chi(E)(y) dy$. Thus for any simple function $r = \sum_{i=1}^n c_i \chi(E_i)$ where $\{E_i\}_{i=1}^n$ are disjoint and $E_i \subset X_j$,

$$\hat{T}_j(\langle r \rangle)(x) = \sum_{i=1}^n c_i T_j(\langle \chi(E_i) \rangle)(x) = \sum_{i=1}^n c_i \mu_{x_j}(E_i) = \int k_j(x, y)r(y) dy$$

for all $x \in X_2$. Thus

$$T(r) = \psi \circ \int k_j(x, y)r(y) dy$$

for any simple function r vanishing outside X_j .

Now let $0 \leq f \in \mathfrak{L}$ and f vanish off of X_j ; then there exists a sequence $\langle f_n \rangle$ of simple functions such that $0 \leq f_n \uparrow f$ and f_n vanishes off of X_j . So

$$T(f) = \lim_{n \rightarrow \infty} T(f_n) = \lim_{n \rightarrow \infty} \psi \circ \int k_j(x, y)f_n(y) dy = \psi \circ \int k_j(x, y)f(y) dy.$$

If we drop the distinction between equivalence classes and functions, we now have for all $f \in \mathfrak{L}$ that vanish off of X_j that $T(f) = \int k_j(x, y)f(y) dy$. The extension from nonnegative f to general f , as usual, uses the decomposition $f = f^+ - f^-$ where $f^+ = f \vee 0$ and $f^- = -f \vee 0$.

If we set $k = \sum_{j=1}^{\infty} k_j$, then $0 \leq k$, and k is $(\mathfrak{A}_2 \times \mathfrak{A}_1)$ measurable. Since $k\chi(x_j \times X_1) = k_j$, $k < \infty$, $\mu_2 \times \mu_1$ a.e. Now let $0 \leq f \in \mathfrak{L}$ and $f = \sum_{j=1}^{\infty} f_j$ where $0 \leq f_j = f \cdot \chi(x_j)$. Since $0 \leq \sum_{j=1}^n f_j \uparrow f$,

$$\begin{aligned} T(f) &= \sum_{j=1}^{\infty} T(f_j) = \sum_{j=1}^{\infty} \int k_j(x, y)f(y) dy \\ &= \sum_{j=1}^{\infty} \int k(x, y)f(y)\chi(x_j)(y) dy = \int k(x, y)f(y) dy. \end{aligned}$$

The extension from nonnegative f to general $f \in \mathfrak{L}$, as before, uses $f = f^+ - f^-$.

In this paragraph we provide an example of a lift of a rather well-known operator. Let (X, \mathfrak{A}, μ) be the usual Lebesgue measure on the real line X , and let L_2 be the square integrable functions defined on X . Clearly L_2 is an ideal of $M(X)$ having an admissible sequence. An integral operator $K: L_2 \rightarrow L_2$, where $K = [k]$ is of Hilbert-Schmidt class when $\iint |k(x, y)|^2 dx dy < \infty$, is [8]. Let u and v be $0 \leq k$ and choose any specific representative

k_1 of k . So k_1 is square integrable and there is a μ null set A such that if $x \notin A$, then $0 \leq \int k_1(x, y)f(y) dy < \infty$. Now define $k_0(x, y) = k_1(x, y)$ when $x \notin A$ and $k_0(x, y) = 0$ when $x \in A$. Thus $k_1 - k_0$ is $\mu \times \mu$ null function and $K = [(k_0)]$. We may now define a lift \hat{K} of K on L_2 by $\hat{K}(f)(x) = \int k_0(x, y)f(y) dy$: i.e. for all $f \in L_2$ and for all $x \in X$, $0 \leq \int k_0(x, y)f(y) dy < \infty$.

BIBLIOGRAPHY

1. W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces*, North-Holland, Amsterdam, 1971.
2. B. Z. Vulih, *Introduction to the theory of partially ordered spaces*, Fizmatgiz, Moscow, 1961; English transl., Noordhoff, Groningen, 1967. MR 24 #A3494; 37 #121.
3. N. Dunford and J. T. Schwartz, *Linear operators. I. General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR 22 #8302.
4. N. Aronszajn and P. Szeptyski, *On general integral transformations*, Math. Ann. 163 (1966), 127–154. MR 32 #8209.
5. J. L. Doob, *Stochastic processes*, Wiley, New York; Chapman & Hall, London, 1953. MR 15, 445.
6. A. Ionescu Tulcea and C. Ionescu Tulcea, *Topics in the theory of lifting*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 48, Springer-Verlag, New York, 1969. MR 43 #2185.
7. W. A. J. Luxemburg and A. C. Zaanen, *The linear modulus of an order bounded linear transformation. II*, Nederl. Akad. Wetensch. Proc. Ser. A75 = Indag. Math. 34 (1972).
8. A. C. Zaanen, *Linear analysis*, North-Holland, Amsterdam, 1964.

DEPARTMENT OF MATHEMATICS, ARYA-MEHR UNIVERSITY OF TECHNOLOGY,
TEHRAN, IRAN