A LATTICE THEORETIC CHARACTERIZATION
OF AN INTEGRAL OPERATOR

LAWRENCE LESSNER

ABSTRACT. We are concerned here with obtaining necessary and sufficient conditions for a linear operator, \( K: \mathcal{L}(X_1, \mathcal{A}_1, \mu_1) \rightarrow \mathcal{M}(X_2, \mathcal{A}_2, \mu_2) \), to be represented by an integral, \( K(f) = \int k(x, y)f(y) \, dy \), with an \( \mathcal{A}_2 \times \mathcal{A}_1 \) measurable kernel \( k(x, y) \). That such conditions are developed in a lattice theoretic context will be shown to be quite natural. Our direction will be to characterize an integral operator by its action pointwise: i.e., \( K(\cdot)(x) \) is a linear functional on a subspace of the essentially bounded functions. Such a development leads one to define the kernel, \( k(x, y) \), in a pointwise fashion also, and as a result we are confronted with the question of the \( \mathcal{A}_2 \times \mathcal{A}_1 \) measurability of \( k(x, y) \).

Definitions and notation. The following definitions, unless otherwise noted, may be found in [1] and [2].

Definition. The real vector space \( R \) is called an ordered vector space when \( R \) is partially ordered by \( \leq \) and satisfies for \( x, y, z \in R \),

1. \( x \leq y \) implies \( x + z \leq y + z \),
2. \( x > 0 \) implies \( rx > 0 \) for any real number \( r > 0 \).

The ordered vector space \( R \) is called a Riesz space when for each \( x, y \in R \) the least upper bound of \( x \) and \( y \), written \( x \vee y \), exists in \( R \). The Riesz space \( (R, \leq) \) is called Dedekind complete if for any subset \( \{x_a\} \subset R \) such that there is an upper bound \( y \in R \) for \( \{x_a\} \), then the least upper bound of \( \{x_a\} \) exists in \( R \): written \( \sup \{x_a\} \in R \). A sequence \( \langle x_n \rangle \) on \( R \) is said to converge in order to \( x \in R \), written \( x_n \rightarrow x(0) \), whenever \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_n' = x \) where

\[
\lim_{n \to \infty} x_n = \sup_{k \geq n} \inf_{k \geq n} x_k, \quad \lim_{n \to \infty} x_n' = \inf_{k \geq n} \sup_{k \geq n} x_k,
\]

and \( \inf \) as usual means greatest lower bound. By \( 0 \leq x_n \downarrow x \) we mean \( 0 \leq x_n \leq x_{n+1} \) and \( \sup_n x_n = x \); \( x_n \downarrow x \) means \( x_n \geq x_{n+1} \) and \( \inf_n x_n = x \). A linear mapping \( T: R_1 \rightarrow R_2 \) between Riesz spaces is called \( (0) \)-continuous when it maps order convergent sequences into order convergent sequences, and \( T \) is called positive when \( x > 0 \) implies \( T(x) > 0 \).


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Let \((X, \mathcal{A}, \mu)\) be a \(\sigma\)-finite measure space and denote by \(M(X)\) the collection of all equivalence classes of \(\mathcal{A}\) measurable finite \(\mu\) a.e. real valued functions on \(X\) modulo \(\mu\) null functions. For \(f, g \in M(X)\) define \(f \leq g\) providing \(f(x) \leq g(x)\) for \(\mu\) a.e. \(x \in X\), then \((M(X), \leq)\) is a Dedekind complete Riesz space: see [1, p. 126] or [3, p. 335]. A linear subspace \(\mathcal{Q} \subset M(X)\) is called an ideal when \(g \in M(X), f \in \mathcal{Q}\) and \(|g| \leq |f|\) implies \(g \in \mathcal{Q}\). We say that a sequence \(\pi = \langle x_j \rangle\) is admissible for \(\mathcal{Q}\) when \(\{x_j\}\) is a countable collection of disjoint, measurable sets of finite measure where \(\bigcup x_j = X\) and \(\chi(x_j) \in \mathcal{Q}\). For a set \(E\), \(\chi(E)\) is the characteristic function of \(E\). We shall be required to distinguish between equivalence classes of functions and functions which are defined and finite everywhere. Let \(\psi\) be the canonical homomorphism that sends a function \(f\), defined and finite real valued a.e., to its equivalence class \(\langle f \rangle\): \(\psi(f) = \langle f \rangle\). For \(S \subset M(X)\), let \(\overline{S} = \{ f / f : X \to (-\infty, \infty) \} \). We may partially order \(M(X)\) as follows: for \(f, g \in \overline{M(X)}\), \(f \leq g\) if and only if for all \(x \in X\), \(f(x) \leq g(x)\). If \(S\) is an ideal of \(M(X)\), then \(\overline{S}\) is an ideal of \(\overline{M(X)}\). Although \(M(X)\) is Dedekind complete, in general \(\overline{M(X)}\) is not Dedekind complete. For \(L\) an ideal of \(M(X), f_n, f \in L\), we have \(f_n \to f(0)\) in \(L\) when there exists \(g \in L\) such that for all \(n\) and all \(x \in X\), \(|f_n(x)| \leq g(x)\) and \(\lim f_n(x) = f(x)\) [2, p. 64].

For the duration of this paper we shall assume that \((X_1, \mathcal{A}_1, \mu_1)\) and \((X_2, \mathcal{A}_2, \mu_2)\) are \(\sigma\)-finite measure spaces with respect to nonnegative, countably additive, extended real valued set functions \(\mu_1\) and \(\mu_2\), respectively. We also assume that \((X_1, \mathcal{A}_1, \mu_1)\) is a separable measure space. We shall denote by \(\mathcal{Q}\) an ideal of \(M(X_1)\) with an admissible sequence \(\pi = \langle x_j \rangle\).

Let \(T : \mathcal{Q} \to M(X_2), S\) a linear subspace of \(\mathcal{Q}\) and \(\hat{T} : S \to M(X_2)\); then \(\hat{T}\) is called a lift of \(T\) on \(S\) when for all \(f \in S, \psi \circ \hat{T}(f) = T(f)\). If a lift \(\hat{T}\) of \(T\) on \(S\) exists, it need not be unique, also \(\hat{T}\) need not inherit even the simplest properties of \(T\): linearity, positivity, order continuity.

A map \(K : \mathcal{Q} \to M(X_2)\) is called an integral operator when there is \(k(x, y) \in M(X_2 \times X_1)\) such that for \(f \in \mathcal{Q}\),
\[
K(f)(x) = \int k(x, y)f(y) \, d\mu_1(y).
\]

\(k\) is called the kernel of \(K\), and the association is denoted \(K = [k]\). That an ideal \(\mathcal{Q}\) of \(M(X_1)\) with an admissible sequence \(\pi\) is a natural domain for an integral operator may be inferred from [4] and [7]. For each \(f \in \mathcal{Q}\) and a.e. \(x \in X_2\) fixed, \(k(x, y)(f(y))\) is integrable on \(X_1\); so for a.e. \(x\),
\[
\int |k(x, y)| f(y) \, d\mu_1(y) < \infty
\]
and, by Fubini’s theorem, defines a function in \(M(X_2)\). Thus \(K : \mathcal{Q} \to M(X_2)\) can be considered the difference of two positive operators \([k^+], [k^-]\): i.e.,
\[
K(f) = \int k^+(x, y)f(y) \, dy - \int k^-(x, y)(f(y)) \, dy
\]
where \([k^\dagger]: \mathcal{L} \to M(X_2)\) are positive operators.

The following theorem gives necessary and sufficient conditions for an operator \(T: \mathcal{L} \to M(X_2)\) to be an integral operator. Since an integral operator is necessarily the difference of two positive operators, it will suffice to consider positive operators only.

**Theorem.** Let \(T: \mathcal{L} \to M(X_2)\) be a positive linear operator; then there exists \(k \in M(X_2 \times X_1)\) such that \(T\) is an integral operator, \(T = [k]\) and \(0 \leq k\) if and only if

1. \(T: \mathcal{L} \to M(X_2)\) is \(0\)-continuous,
2. for each \(j\) there is a lift \(\hat{T}_j\) of \(T\) on \(L_\infty(x_j)\) that is positive, linear and order continuous.

**Proof.** Let \(T = [k]\) be a positive linear integral operator with kernel \(0 \leq k\). By [2, p. 215], it is sufficient to show that if \(f_n \in \mathcal{L}\) and \(0 \leq f_n \downarrow 0\), then \(0 \leq T(f_n) \downarrow 0\), to obtain order continuity for \(T\). This follows easily from the Lebesgue dominated convergence theorem. To verify (2) let \(k_0\) be a specific kernel: \(k_0 \in k \cap M(X_2 \times X_1)\) such that \(0 \leq k_0(x, y)\) for every \((x, y) \in X_2 \times X_1\). For each \(j\), \(\int k_0(x, y) \chi(x_j)(y)\, dy < \infty\) except for \(x \in A_j\), where \(\mu_2(A_j) = 0\). Now take \(k_1(x, y) = k_0(x, y)\) for \((x, y) \in \bigcup A_j \times X_1\) and define \(k_1(x, y) = 0\) for \((x, y) \in \mathcal{L} \times X_1\). Thus \(k_1\) and \(k_0\) differ on a set of \(\mu_2 \times \mu_1\) measure zero. If \(f \in L_\infty(x_j)\), then for some \(0 < c\) we have \(|f| \leq c\chi(x_j)\).

Consequently, \[
\left|\int k_1(x, y)\, dy\right| \leq \int k_1(x, y) \cdot c\chi(x_j)(y)\, dy < \infty
\]
for all \(x \in X_2\). For \(f \in L_\infty(x_j)\), define

\[
\hat{T}_j(f) = \int k_1(x, y)(y)\, dy;
\]
then \(\hat{T}_j\) is a lift of \(T\) on \(L_\infty(x_j)\). It is obvious that \(\hat{T}_j\) is positive and linear. If \(f_n \in L_\infty(x_j)\), \(0 \leq f_n \downarrow 0\), then a simple application of Lebesgue’s dominated convergence theorem shows that \(0 \leq \hat{T}_j(f_n)(x) \downarrow 0\) for all \(x\) and \(\hat{T}_j\) is order continuous.

Now let us suppose that \(T: \mathcal{L} \to M(X_2)\) is a positive, linear and order continuous map that satisfies (2). We shall construct the integral representation for \(T\) by viewing \(\hat{T}_j(x)\) as a measure on the relativized space \((X_j, \bar{\mathcal{G}}_j \cap X_j, \mu_j)\) and applying the Radon-Nikodym theorem. Let \(\bar{\mathcal{G}}_j = \{E \cap X_j : E \in \bar{\mathcal{G}}_j\}\) and \(x \in X_2\) be fixed, then define for \(E \in \bar{\mathcal{G}}_j\), \(\mu_{x_j}(E) = \hat{T}_j(\chi(E))(x)\). The finite additivity of \(\mu_{x_j}\) follows from the linearity of \(\hat{T}_j\), and the nonnegativity of \(\mu_{x_j}\) comes from the positivity of \(\hat{T}_j\). Let \(\langle E_i \rangle\) be a countable disjoint sequence from \(\bar{\mathcal{G}}_j\); then \(\chi(\bigcup_{i=1}^\infty E_i) = \chi(\bigcup_{i=1}^\infty E_i) = f_n\) and \(0 \leq f_n \downarrow 0\). Since \(\hat{T}_j\) is order continuous, \(0 \leq \hat{T}_j(f_n(x)) \downarrow 0\) for all \(x \in X_2\). Consequently, \(\lim_{n \to \infty} \mu_{x_j}(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu_{x_j}(E_i) = 0\); \(\mu_{x_j}\) is countably additive.
Now if \( E \in \mathcal{E}_j \) is a \( \mu_1 \) null set, then
\[
\hat{T}_j((\chi(E))(x)) = \hat{T}_j(2(\chi(E)))(x) = 2\hat{T}_j((\chi(E))(x))
\]
for all \( x \in X \), so \( \hat{T}_j((\chi(E))(x)) = 0 \) for all \( x \in X \) and \( \mu_{X_j} \) is absolutely continuous with respect to \( \mu_1 \). Let \( k_j(x, y) \) be the Radon-Nikodym derivative of \( \mu_{X_j} \) with respect to \( \mu_1 \). Clearly \( 0 \leq k_j(x, y) \) for all \( x \) and \( y \). Since \( (X_1, \mathcal{E}_1, \mu_1) \) is separable, it follows by [5, p. 616], that \( k_j(x, y) \) is \( \mathcal{E}_2 \times \mathcal{E}_1 \) measurable.

For \( E \in \mathcal{E}_j \), we have \( \mu_{X_j}(E) = \int k_j(x, y)\chi(E)(y) \, dy \). Thus for any simple function \( r = \sum_{i=1}^{n} c_i \chi(E_i) \) where \( \{E_i\}_{i=1}^{n} \) are disjoint and \( E_i \subset X_j \),
\[
\hat{T}_j((r))(x) = \sum_{i=1}^{n} c_i \hat{T}_j((\chi(E_i))(x)) = \sum_{i=1}^{n} c_i \mu_{X_j}(E_i) = \int k_j(x, y)r(y) \, dy
\]
for all \( x \in X \). Thus
\[
T(r) = \int k_j(x, y)r(y) \, dy
\]
for any simple function \( r \) vanishing outside \( X_j \).

Now let \( 0 \leq f \in L^1 \) and \( f \) vanish off of \( X_j \); then there exists a sequence \( \langle f_n \rangle \) of simple functions such that \( 0 \leq f_n \uparrow f \) and \( f_n \) vanishes off of \( X_j \). So
\[
T(f) = \lim_{n \to \infty} T(f_n) = \lim_{n \to \infty} \int k_j(x, y)f_n(y) \, dy = \int k_j(x, y)f(y) \, dy.
\]
If we drop the distinction between equivalence classes and functions, we now have for all \( f \in L^1 \) that vanish off of \( X_j \), that \( T(f) = \int k_j(x, y)f(y) \, dy \). The extension from nonnegative \( f \) to general \( f \), as usual, uses the decomposition \( f = f^+ - f^- \) where \( f^+ = f \vee 0 \) and \( f^- = -f \vee 0 \).

If we set \( k = \sum_{j=1}^{\infty} k_j \), then \( 0 \leq k \), and \( k \) is \( \mathcal{E}_2 \times \mathcal{E}_1 \) measurable. Since \( k\chi(x_j \times X_1) = k_j \cdot k_j \), \( k \cdot k_j \). \( \mu_2 \times \mu_1 \) a.e. Now let \( 0 \leq f \in L^1 \) and \( f = \sum_{j=1}^{\infty} f_j \), where \( 0 \leq f_j = f \cdot \chi_j \). Since \( 0 \leq \sum_{j=1}^{\infty} f_j \),
\[
T(f) = \sum_{j=1}^{\infty} T(f_j) = \sum_{j=1}^{\infty} \int k_j(x, y)f(y) \, dy
\]
\[
= \sum_{j=1}^{\infty} \int k(x, y)f(y)\chi_j(y) \, dy = \int k(x, y)f(y) \, dy.
\]

The extension from nonnegative \( f \) to general \( f \in L^1 \), as before, uses \( f = f^+ - f^- \).

In this paragraph we provide an example of a lift of a rather well-known operator. Let \( (X, \mathcal{E}, \mu) \) be the usual Lebesgue measure on the real line \( X \), and let \( L_2 \) be the square integrable functions defined on \( X \). Clearly \( L_2 \) is an ideal of \( M(X) \) having an admissible sequence. An integral operator \( K: L_2 \to L_2 \) where \( K = [k] \) is of Hilbert-Schmidt class when \( \iint |k(x, y)|^2 \, dx \, dy < \infty \); see [8]. Let us suppose \( 0 \leq k \), and choose any specific representative
k_1 of k. So k_1 is square integrable and there is a \( \mu \) null set \( A \) such that if \( x \not\in A \), then \( 0 \leq \int k_1(x, y)f(y) \, dy < \infty \). Now define \( k_0(x, y) = k_1(x, y) \) when \( x \not\in A \) and \( k_0(x, y) = 0 \) when \( x \in A \). Thus \( k_1 - k_0 \) is \( \mu \times \mu \) null function and \( K = \{(k_0)\} \). We may now define a lift \( \tilde{K} \) of \( K \) on \( L_2 \) by \( \tilde{K}(f)(x) = \int k_0(x, y)f(y) \, dy \): i.e. for all \( f \in L_2 \) and for all \( x \in X \), \( 0 \leq \int k_0(x, y)f(y) \, dy < \infty \).

**BIBLIOGRAPHY**


DEPARTMENT OF MATHEMATICS, ARYA-MEHR UNIVERSITY OF TECHNOLOGY, TEHRAN, IRAN