PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

TAUBERIAN CONCLUSIONS

K. A. JUKES AND I. J. MADDOX

For Professor B. Kuttner on his retirement

ABSTRACT. Littlewood's celebrated Tauberian theorem states that $\Sigma a_n = s$ (Abel) and $na_n = O(1)$ imply $s_n = \Sigma_{k=1}^n a_k$ converges to s, the Tauberian condition $na_n = O(1)$ being best possible. We investigate 'best possibility' of the conclusion $s_n - s = o(1)$, replacing the usual Tauberian condition by $(q_n a_n) \in E$ where (q_n) is a positive sequence and E a given sequence space.

1. **Introduction.** The first Tauberian theorem of O type was proved by Hardy [1]. It asserts that $\Sigma a_n = s$ (C, k) and $na_n = O(1)$ imply $s_n = \Sigma_{k=1}^n a_k$ converges to s. Hardy raised the question as to whether (C, k) summability could be replaced by Abel summability, stating he was inclined to think it could not. However, Littlewood, in his now famous paper [3], showed that $\Sigma a_n = s$ (Abel) and $na_n = O(1)$ implied $s_n \to s$, and also, that the big O condition was best possible in that if $0 < \phi(n) \to \infty$, there exists a divergent Σa_n such that $n|a_n| < \phi(n)$ and Σa_n is Abel-summable.

In the present note we consider the 'best possibility' of certain Tauberian conclusions. One of the simplest questions is to ask whether $\Sigma a_n = 0$ (Abel) and $na_n = O(1)$ imply that $s_n = o(1)$ is best possible. That this is so is a special case of Corollary 1 below.

Let $q = (q_n)$ and $p = (p_n)$ denote sequences of positive real numbers. Writing $a = (a_n)$ and $qa = (q_n a_n)$, for a given sequence space E let $S(q, E) = \{a\colon qa \in E \text{ and } s_n \to 0\}$. Denote by $P(q, E)$ the property that for all $p \notin l_\infty$, $\exists a \in S(q, E)$ such that $p_n s_n \not\to 0$. Our main aim is to establish results of the form: $P(q, E)$ if and only if $1/q = (1/q_n) \notin E'$, where E' is another sequence space which is, in a sense, dual to E.

In the following table we list corresponding spaces E, E'. Here $\gamma = \{a\colon \Sigma a_n \text{ converges}\}$ and $BV_0 = \{a\colon \Sigma|\Delta a_n| < \infty \text{ and } a_n \to 0\}$ where $\Delta a_n = a_n - a_{n+1}$.

2. **Verification of the table.** In proving the results in the table we shall use the negation of $P(q, E)$, which we denote by $Q(q, E)$, namely $\exists p \notin l_\infty$ such that for all $a \in S(q, E)$, $p_n s_n \to 0$.

(i) To prove $P(q, l_r)$ implies $1/q \notin c_0$ $(0 < r \leq 1)$, we show equivalently

Received by the editors October 9, 1974.

AMS (MOS) subject classifications (1970). Primary 40E05; Secondary 40G10, 40G99.

Key words and phrases. Tauberian theorems, best possible conclusions, Abel, Ingham methods.

Copyright © 1975, American Mathematical Society

E	E'
l_r $(0 < r \leq 1)$	c_0
l_r $(1 < r < \infty)$	l_s $(1/r + 1/s = 1)$
l_∞	l_1
c_0	l_1
γ	BV_0

that $1/q \in c_0$ implies $Q(q, l_r)$. If $1/q \in c_0$, we may choose p_n so that $1/p_n = \sup_{k \geq n+1} 1/q_k$. For then, if $a \in S(q, l_r)$, given $\epsilon > 0$,

$$|s_n|^r = \left|\sum_{n+1}^{\infty} a_k\right|^r \leq \left(\frac{1}{p_n}\right)^r \sum_{n+1}^{\infty} |q_k a_k|^r < \epsilon\left(\frac{1}{p_n}\right)^r \qquad (n > n_0, \text{ say}),$$

on using b, $c \geq 0$ implies $(b + c)^r \leq b^r + c^r$ (see e.g. Maddox [4, p.22]).

Conversely, suppose $1/q \notin c_0$ and $p \notin l_\infty$. Then there exists $c > 0$ and (m_i) strictly increasing such that $q_{m_i} \equiv q(m_i) \leq c$ (each i). Choose $k_1 > 1$ such that $p(k_1) > 2$ and then $m_i > k_1$ such that $q(m_i) \leq c$. Denote m_i by t_1. Let

$$a_k = \begin{cases} 0 & (1 \leq k < t_1), \\ 1/p(k_1) & (k = t_1). \end{cases}$$

For $n > 1$ choose $k_n > t_{n-1}$ such that $p(k_n) > 2p(k_{n-1})$ and then $m_j > k_n$ such that $q(m_j) \leq c$. Denote m_j by t_n. Let

$$a_k = \begin{cases} 0 & (t_{n-1} < k < t_n), \\ 1/p(k_n) - 1/p(k_{n-1}) & (k = t_n). \end{cases}$$

Then s_n decreases $(n > t_1)$, $s(t_n) = 1/p(k_n) < 1/2^n$ and for $n > 1$, $s(k_n) = 1/p(k_{n-1}) > 2/p(k_n)$. Further for $n \geq 2$,

$$|q(t_n)(1/p(k_n) - 1/p(k_{n-1}))| \leq c/p(k_{n-1}) \leq c/2^{n-1},$$

whence $qa \in l_r$, and so $P(q, l_r)$ holds.

(ii) Suppose $a \in S(q, l_r)$ $(1 < r < \infty)$ and $1/q \in l_s$ $(1/r + 1/s = 1)$. Taking $p_n = (\Sigma_{n+1}^{\infty} 1/q_k^s)^{-1/s}$, $Q(q, l_r)$ holds. For by Hölder's inequality, given $\epsilon > 0$,

$$|s_n| = \left|\sum_{n+1}^{\infty} a_k\right| \leq \left(\sum_{n+1}^{\infty} |q_k a_k|^r\right)^{1/r} \Big/ p_n < \frac{\epsilon}{p_n} \qquad (n > n_0, \text{ say}).$$

Conversely, let $1/q \notin l_s$ and $p \notin l_\infty$. Write

$$M(m, n) = \sum_{m+1}^{n} \frac{1}{q_k^s} \qquad (n \geq m+1).$$

Put $a_1 = 0$. Choose $n_1 > 1$ so that $M(1, n_1) > 1$ and $p(n_1) > 1$. Put

$$a_k = 1/M(1, n_1)p(n_1)q_k^s \qquad (2 \leq k \leq n_1).$$

For $i \geq 1$ choose $n_{i+1} > n_i$ so that $M(n_i, n_{i+1}) > 1$ and $p(n_{i+1}) > 2p(n_i)$. Put

$$a_k = \frac{(1/p(n_{i+1}) - 1/p(n_i))}{M(n_i, n_{i+1})q_k^s} \qquad (n_i < k \leq n_{i+1}).$$

Then s_n decreases $(n > n_1)$ and $s(n_{i+1}) = 1/p(n_{i+1})$. Further

$$\sum_{n_i+1}^{n_{i+1}} |q_k a_k|^r \leq \sum_{n_i+1}^{n_{i+1}} 1/M^r(n_i, n_{i+1})q_k^{r(s-1)}p^r(n_i)$$

$$= 1/M^{r-1}(n_i, n_{i+1})p^r(n_i) < 1/p^r(n_i) < 1/2^{r(i-1)},$$

whence $qa \in l_r$.

(iii) Suppose $a \in S(q, l_\infty)$ and $1/q \in l_1$. Taking $p_n = (\Sigma_{n+1}^{\infty} 1/q_k)^{-1/2}$, we have $p_n \to \infty$ and $|p_n s_n| \leq (\sup_n |q_n a_n|)p_n^{-1/2} \to 0$, and so $Q(q, l_\infty)$ holds.

Conversely, let $1/q \notin l_1$ and $p \notin l_\infty$. Let a be as constructed in (ii) above, taking $s = 1$. Then again s_n decreases $(n > n_1)$ and $s(n_{i+1}) = 1/p(n_{i+1})$. Further for $n_i < k \leq n_{i+1}$, $|q_k a_k| \leq 1/p(n_i) \to 0$. Hence $P(q, l_\infty)$ holds.

(iv) Suppose $1/q \in l_1$. Then $Q(q, c_0)$ trivially holds with $1/p_n = \Sigma_{n+1}^{\infty} 1/q_k$. Conversely, if $1/q \notin l_1$, then $P(q, c_0)$ holds with a as in (iii).

(v) Suppose $a \in S(q, \gamma)$ and $1/q \in BV_0$. Write $Q_n = \Sigma_{n+1}^{\infty} |\Delta(1/q_k)|$. For $m \geq n+2$,

$$\sum_{k=n+1}^{m} a_k = \frac{1}{q_m} \sum_{k=n+1}^{m} q_k a_k + \sum_{k=n+1}^{m-1} \left(\sum_{r=n+1}^{k} q_r a_r \right) \Delta\left(\frac{1}{q_k}\right).$$

So given $\epsilon > 0$, for $n > n_0(\epsilon)$,

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon \left(\frac{1}{q_m} + \sum_{n+1}^{m-1} \left| \Delta\left(\frac{1}{q_k}\right) \right| \right),$$

whence

$$|s_n| = \left| \sum_{n+1}^{\infty} a_k \right| \leq \epsilon Q_n.$$

Note that $Q_n = 0$ implies $q_k = q_{n+1}$ $(k > n)$, contradicting $1/q \in c_0$. So $Q(q, \gamma)$ holds with $p_n = 1/Q_n$.

Conversely suppose $1/q \notin BV_0$ and $p \notin l_\infty$. Either

(a) $\Sigma|\Delta(1/q_k)| < \infty$ and $1/q_k \not\to 0$, or

(b) $\Sigma|\Delta(1/q_k)| = \infty$.

Now $\Sigma|\Delta(1/q_k)| < \infty$ implies $1/q \in c$ and so (a) implies $q \in l_\infty$. Further, (a) implies $1/q \notin l_1$. But the sequence a constructed in (iii) satisfies $a \in l_1$. Since $q \in l_\infty$, then $qa \in l_1$. In particular (a) implies $P(q, \gamma)$.

When (b) holds, first suppose $1/q \notin c_0$. Then $P(q, l_r)$ $(0 < r \leq 1)$ holds But $qa \in l_r$ implies $qa \in \gamma$ and so $P(q, \gamma)$ holds. Now suppose $1/q \in c_0$. Writing $t_n = \Sigma_{k=1}^n q_k a_k$, we have

$$s_n = \sum_1^n a_k = \frac{t_n}{q_n} + \sum_{k=1}^{n-1} t_k \Delta\left(\frac{1}{q_k}\right) \qquad (n \geq 2).$$

Define $\operatorname{sgn} z = |z|/z$ $(z \neq 0)$, $\operatorname{sgn} 0 = 0$. Choose $n_1 > 2$ such that $p(n_1) > 1$ and $\Sigma_1^{n_1 - 1} |\Delta(1/q_k)| > 1$. Define

$$t_k = \begin{cases} (\operatorname{sgn} \Delta(1/q_k))/p(n_1)\Sigma_1^{n_1-1} |\Delta(1/q_k)| & (1 \leq k < n_1), \\ 0 \quad (k = n_1). \end{cases}$$

For $r \geq 1$, choose $n_{r+1} > n_r$ such that $\Sigma_{n_r+1}^{n_{r+1}-1} |\Delta(1/q_k)| > 1$ and $p(n_{r+1}) > 2p(n_r)$. Define

$$t_k = \begin{cases} (\operatorname{sgn} \Delta(1/q_k))(1/p(n_{r+1}) - 1/p(n_r))/\Sigma_{n_r+1}^{n_{r+1}-1} |\Delta(1/q_k)| & (n_r < k < n_{r+1}), \\ 0 \quad (k = n_{r+1}). \end{cases}$$

Then $s(n_{r+1}) = 1/p(n_{r+1})$, $t_n \to 0$, $t_n/q_n \to 0$ and $\Sigma_1^{n-1} t_k \Delta(1/q_k) \to 0$. Hence $P(q, \gamma)$ holds, completing the table.

For corresponding spaces E, E' we immediately have

Corollary 1. *Suppose* $\Sigma a_n = 0(A)$ *and* $qa \in E$ *imply* $s_n = o(1)$, *where* A *is a regular summability method. If* $1/q \notin E'$, *then the conclusion* $s_n = o(1)$ *is best possible.*

If we take A to be the Abel method, $q = (n)$ and $E = l_\infty$ in Corollary 1, then we have the answer to our question in the introduction regarding the best possible nature of $s_n = o(1)$ in the classical Tauberian case.

A later Tauberian theorem of Ingham's [2, Theorem 2] states that $\Sigma a_n = s\ (I)$ and $na_n = O(1)$ imply $s_n \to s$, where (I) denotes Ingham summability. Taking $a_n = \mu(n)/n$, where $\mu(n)$ is the Möbius function, gives the prime number theorem. Although (I) is not regular, for $E = l_r$ $(0 < r \leq \infty)$ or c_0, we do have

Corollary 2. *Suppose* $\Sigma a_n = 0(I)$ *and* $qa \in E$ *imply* $s_n = o(1)$. *If* $1/q \notin E'$, *then the conclusion* $s_n = o(1)$ *is best possible.*

The proof follows immediately, using the following lemma, since in (i) – (iv) above each constructed $(a_k) \in l_1$.

Lemma 1. $\Sigma|a_k| < \infty$ *implies* Σa_k *is summable* (I).

Proof.

$$I_n \equiv \frac{1}{n}\sum_{k=1}^{n} ka_k\left[\frac{n}{k}\right] = \sum_{k=1}^{n} a_k + O\left(\frac{1}{n}\sum_{k=1}^{n} k|a_k|\right).$$

But $\Sigma|a_k| < \infty$ implies $k|a_k| \to 0(C, 1)$. Hence $I_n = s_n + o(1)$.

3. **Best possibility when** $1/q \in E'$. If $1/q \in E'$, then $a \in S(q, E)$ does not imply that $s_n = o(1)$ is best possible. It is interesting to consider what is best possible in this case. As an illustration we have, when $E = c_0$, the following

Theorem. *Suppose* $1/q \in l_1$. *Write* $R(n) = \Sigma_{n+1}^{\infty} 1/q_k$. *Then*:
(a) $a \in S(q, c_0)$ *implies* $s_n = o(R(n))$.
(b) *For all* $p \notin l_\infty$, $\exists a \in S(q, c_0)$ *such that* $p_n s_n/R(n) \not\to 0$.

Proof. (a) was noted in (iv) above.

(b) Write $M(m, n) = \Sigma_{m+1}^{n} 1/q_k$ $(n \geq m+1)$. Choose $n_1 > 1$ so that $p(n_1) > 1$. Define $a_k = R(n_1)/M(0, n_1)p(n_1)q_k$ $(1 \leq k \leq n_1)$. For $r \geq 1$ choose $n_{r+1} > n_r$ so that $R(n_r)/M(n_r, n_{r+1}) < 2$ and $p(n_{r+1}) > 2p(n_r)$. Define

$$a_k = \frac{(R(n_{r+1})/p(n_{r+1}) - R(n_r)/p(n_r))}{M(n_r, n_{r+1})q_k}. \qquad (n_r < k \leq n_{r+1}).$$

Then, for $n_r < k \leq n_{r+1}$, $|q_k a_k| \leq 2/p(n_r) \to 0$, while s_n decreases $(n > n_1)$ and $s(n_r) = R(n_r)/p(n_r)$, completing the proof.

Finally, we have

Corollary 3. *Suppose* $1/q \in l_1$.
(a) *If* $qa \in c_0$ *and* $\Sigma a_n = 0(I)$, *then* $s_n = o(R(n))$.
(b) *For all* $p \notin l_\infty$, $\exists a$ *such that* $qa \in c_0$, $\Sigma a_n = 0(I)$ *but* $p_n s_n/R(n) \not\to 0$.

This corollary is an immediate consequence of the above Theorem since one can easily establish

Lemma 2. *If* $1/q \in l_1$, *then* $qa \in c_0$ *implies* $s_n - I_n \to 0$.

REFERENCES

1. G. H. Hardy, *Theorems relating to the summability and convergence of slowly oscillating series*, Proc. London Math. Soc (2) 8 (1910), 301–320.

2. A. E. Ingham, *Some Tauberian theorems connected with the prime number theorem*, J. London Math. Soc. 20 (1945), 171–180. MR 8, 147.

3. J. E. Littlewood, *The converse of Abel's theorem on power series*, Proc. London Math. Soc. (2) 10 (1910/11), 434–448.

4. I. J. Maddox, *Elements of functional analysis*, Cambridge, 1970.

DEPARTMENT OF PURE MATHEMATICS, QUEEN'S UNIVERSITY, BELFAST, UNITED KINGDOM (Current address of I. J. Maddox)

Current address (Dr. K. A. Jukes): Department of Mathematics and Statistics, Sheffield Polytechnic, Sheffield, England