TAUBERIAN CONCLUSIONS

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For Professor B. Kuttner on his retirement

ABSTRACT. Littlewood's celebrated Tauberian theorem states that \( \Sigma a_n = s \) (Abel) and \( na_n = O(1) \) imply \( s = \Sigma_{k=1}^{\infty} a_k \) converges to \( s \), the Tauberian condition \( na_n = O(1) \) being best possible. We investigate 'best possibility' of the conclusion \( s_n - s = o(1) \), replacing the usual Tauberian condition by \( (q_n a_n) \in E \) where \( (q_n) \) is a positive sequence and \( E \) a given sequence space.

1. Introduction. The first Tauberian theorem of \( O \) type was proved by Hardy [1]. It asserts that \( \Sigma a_n = s \) \((C, k)\) and \( na_n = O(1) \) imply \( s = \Sigma_{k=1}^{\infty} a_k \) converges to \( s \). Hardy raised the question as to whether \((C, k)\) summability could be replaced by Abel summability, stating he was inclined to think it could not. However, Littlewood, in his now famous paper [3], showed that \( \Sigma a_n = s \) (Abel) and \( na_n = O(1) \) implied \( s = O(\infty) \), also, that the big \( O \) condition was best possible in that if \( 0 < \phi(n) \to \infty \), there exists a divergent \( \Sigma a_n \) such that \( n|a_n| < \phi(n) \) and \( \Sigma a_n \) is Abel-summable.

In the present note we consider the 'best possibility' of certain Tauberian conclusions. One of the simplest questions is to ask whether \( \Sigma a_n = 0 \) (Abel) and \( na_n = O(1) \) imply that \( s_n = o(1) \) is best possible. That this is so is a special case of Corollary 1 below.

Let \( q = (q_n) \) and \( p = (p_n) \) denote sequences of positive real numbers. Writing \( a = (a_n) \) and \( qa = (q_n a_n) \), for a given sequence space \( E \) let \( S(q, E) = \{a: qa \in E \text{ and } s_n \to 0\} \). Denote by \( P(q, E) \) the property that for all \( p \not\leq l_\infty \), \( 3a \in S(q, E) \) such that \( p s_n \to 0 \). Our main aim is to establish results of the form: \( P(q, E) \) if and only if \( l/a = (l/a_n) \in E' \), where \( E' \) is another sequence space which is, in a sense, dual to \( E \).

In the following table we list corresponding spaces \( E, E' \). Here \( \gamma = \{|a: \Sigma a_n \text{ converges}\} \) and \( BV_0 = \{|a: \Sigma |\Delta a_n| < \infty \text{ and } a_n \to 0\} \) where \( \Delta a_n = a_n - a_{n+1} \).

2. Verification of the table. In proving the results in the table we shall use the negation of \( P(q, E) \), which we denote by \( Q(q, E) \), namely \( 3p \not\leq l_\infty \) such that for all \( a \in S(q, E), p s_n \to 0 \).

(i) To prove \( P(q, l_r) \) implies \( 1/q \not\leq c_0 \) \((0 < r \leq 1)\), we show equivalently

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that $1/q \in c_0$ implies $Q(q, l_r)$. If $1/q \in c_0$, we may choose $p_n$ so that 
$1/p_n = \sup_{k \geq n+1} 1/q_k$. For then, if $a \in S(q, l_r)$, given $\epsilon > 0$,

$$|s_n|^r = \sum_{k \geq n+1} |a_k|^r \leq \left( \frac{1}{p_n} \right)^r \sum_{k \geq n+1} |q_k a_k|^r < \epsilon \left( \frac{1}{p_n} \right)^r \quad (n > n_0, \text{say}),$$

on using $b, c \geq 0$ implies $(b+c)^r \leq b^r + c^r$ (see e.g. Maddox [4, p.22]). Conversely, suppose $1/q \notin c_0$ and $p \notin l_\infty$. Then there exists $c > 0$ and $(m_i)$ strictly increasing such that $q_{m_i} = q(m_i) \leq c$ (each $i$). Choose $k_1 > 1$ such that $p(k_1) > 2$ and then $m_i > k_i$ such that $q(m_i) \leq c$. Denote $m_i$ by $t_1$. Let

$$a_k = \begin{cases} 0 & (1 \leq k < t_1), \\ 1/p(k_1) & (k = t_1). \end{cases}$$

For $n > 1$ choose $k_n > t_{n-1}$ such that $p(k_n) > 2p(k_{n-1})$ and then $m_j > k_n$ such that $q(m_j) \leq c$. Denote $m_j$ by $t_n$. Let

$$a_k = \begin{cases} 0 & (t_{n-1} < k < t_n), \\ 1/p(k_n) - 1/p(k_{n-1}) & (k = t_n). \end{cases}$$

Then $s_n$ decreases $(n > t_1)$, $s(t_n) = 1/p(k_n) < 1/2^n$ and for $n > 1$, $s(k_n) = 1/p(k_{n-1}) > 2/p(k_n)$. Further for $n \geq 2$,

$$|q(t_n)(1/p(k_n) - 1/p(k_{n-1}))| \leq c/p(k_{n-1}) \leq c/2^{n-1},$$

whence $qa \in l_r$, and so $P(q, l_r)$ holds.

(ii) Suppose $a \in S(q, l_r)$ $(1 < r < \infty)$ and $1/q \in l_s$ $(1/r + 1/s = 1)$. Taking $p_n = (\sum_{k=1}^\infty 1/q_k)^{-1/s}$, $Q(q, l_r)$ holds. For by Hölder's inequality, given $\epsilon > 0$,

$$|s_n|^r = \sum_{k \geq n+1} |a_k|^r \leq \left( \sum_{k \geq n+1} |q_k a_k|^r \right)^{1/r} p_n^{1/r} < \epsilon \left( \frac{1}{p_n} \right)^{1/r} \quad (n > n_0, \text{say}).$$

Conversely, let $1/q \notin l_s$ and $p \notin l_\infty$. Write
$M(m, n) = \sum_{m+1}^{n} \frac{1}{q_{k}^{s}} \quad (n \geq m + 1).$

Put $a_{1} = 0$. Choose $n_{1} > 1$ so that $M(1, n_{1}) > 1$ and $p(n_{1}) > 1$. Put

$$a_{k} = 1/M(1, n_{1})p(n_{1})q_{k}^{s} \quad (2 \leq k \leq n_{1}).$$

For $i \geq 1$ choose $n_{i+1} > n_{i}$ so that $M(n_{i}, n_{i+1}) > 1$ and $p(n_{i+1}) > 2p(n_{i})$. Put

$$a_{k} = \frac{1/p(n_{i+1}) - 1/p(n_{i})}{M(n_{i}, n_{i+1})q_{k}^{s}} \quad (n_{i} < k \leq n_{i+1}).$$

Then $s_{n}$ decreases $(n > n_{1})$ and $s(n_{i+1}) = 1/p(n_{i+1})$. Further

$$\sum_{n_{i}+1}^{n_{i+1}} \frac{|q_{k}a_{k}|}{r} \leq \sum_{n_{i}+1}^{n_{i+1}} 1/M(n_{i}, n_{i+1})q_{k}^{r(s-1)}p^{r}(n_{i})$$

$$= 1/M^{-1}(n_{i}, n_{i+1})p^{r}(n_{i}) < 1/p(n_{i}) < 1/2^{r(i-1)},$$

whence $qa \in l_{r}$.

(iii) Suppose $a S(q, l_{\infty})$ and $1/q \in l_{1}$. Taking $p_{n} = (\Sigma_{n+1}^{\infty} 1/q_{k})^{-1/2}$, we have $p_{n} \to \infty$ and $|p_{n}s_{n}| \leq (\sup_{n} |q_{n}a_{n}|)p_{n}^{-1/2} \to 0$, and so $Q(q, l_{\infty})$ holds.

Conversely, let $1/q \notin l_{1}$ and $p \notin l_{\infty}$. Let $a$ be as constructed in (ii) above, taking $s = 1$. Then again $s_{n}$ decreases $(n > n_{1})$ and $s(n_{i+1}) = 1/p(n_{i+1})$. Further for $n_{i} < k \leq n_{i+1}$, $|q_{k}a_{k}| < 1/p(n_{i}) \to 0$. Hence $P(q, l_{\infty})$ holds.

(iv) Suppose $1/q \in l_{1}$. Then $Q(q, c_{0})$ trivially holds with $1/p_{n} = \Sigma_{n+1}^{\infty} 1/q_{k}$. Conversely, if $1/q \notin l_{1}$, then $P(q, c_{0})$ holds with $a$ as in (iii).

(v) Suppose $a S(q, \gamma)$ and $1/q \in BV_{0}$. Write $Q_{n} = \Sigma_{n+1}^{\infty} |\Lambda(1/q_{k})|$. For $m \geq n + 2,$

$$\frac{m}{m+1} \sum_{k=n+1}^{m} a_{k} = \frac{1}{q_{m}} \sum_{k=n+1}^{m} q_{k}a_{k} + \sum_{k=n+1}^{m-1} \left( \sum_{r=n+1}^{k} q_{r}a_{r} \right) \Lambda \left( \frac{1}{q_{k}} \right).$$

So given $\epsilon > 0$, for $n > n_{0}(\epsilon),$

$$\left| \sum_{k=n+1}^{m} a_{k} \right| < \epsilon \left( \frac{1}{q_{m}} \sum_{n+1}^{m-1} \left| \Lambda \left( \frac{1}{q_{k}} \right) \right| \right),$$

whence

$$|s_{n}| = \left| \sum_{n+1}^{\infty} a_{k} \right| \leq \epsilon Q_{n}.$$
Now $\sum |\Lambda(1/q_k)| < \infty$ implies $1/q \in c$ and so (a) implies $q \in l_\infty$. Further, (a) implies $1/q \notin l_1$. But the sequence $a$ constructed in (iii) satisfies $a \in l_1$. Since $q \in l_\infty$, then $qa \in l_1$. In particular (a) implies $P(q, y)$.

When (b) holds, first suppose $1/q \notin c_0$. Then $P(q, l_r) (0 < r \leq 1)$ holds. But $qa \in l_r$ implies $qa \in y$ and so $P(q, y)$ holds. Now suppose $1/q \in c_0$.

Writing $t_n = \sum_{k=1}^{n} q_k a_k$, we have

$$s_n = \sum_{k=1}^{n} a_k - \frac{t_n}{q_n} + \sum_{k=1}^{n-1} t_k \Lambda\left(\frac{1}{q_k}\right) \quad (n \geq 2).$$

Define $\text{sgn} z = |z|/z$ ($z \neq 0$), $\text{sgn} 0 = 0$. Choose $n_1 > 2$ such that $p(n_1) > 1$ and $\sum_{1}^{n_1-1} |\Lambda(1/q_k)| > 1$. Define

$$t_k = \begin{cases} (\text{sgn} \Lambda(1/q_k)) \cdot \frac{1}{p(n_k)} \sum_{1}^{n_k-1} |\Lambda(1/q_k)| & (1 \leq k < n_1), \\ 0 & (k = n_1). \end{cases}$$

For $r \geq 1$, choose $n_{r+1} > n_r$ such that $\sum_{n_r+1}^{n_{r+1}} |\Lambda(1/q_k)| > 1$ and $p(n_{r+1}) > 2p(n_r)$. Define

$$t_k = \begin{cases} (\text{sgn} \Lambda(1/q_k)) \cdot \frac{1}{p(n_{r+1})} - 1/p(n_r) / \sum_{n_r+1}^{n_{r+1}} |\Lambda(1/q_k)| & (n_r < k < n_{r+1}), \\ 0 & (k = n_{r+1}). \end{cases}$$

Then $s(n_{r+1}) = 1/p(n_{r+1})$, $t_n \to 0$, $t_n/q_n \to 0$ and $\sum_{1}^{n-1} t_k \Lambda(1/q_k) \to 0$.

Hence $P(q, y)$ holds, completing the table.

For corresponding spaces $E, E'$ we immediately have

**Corollary 1.** Suppose $\Sigma a_n = 0(\Lambda)$ and $qa \in E$ imply $s_n = o(1)$, where $\Lambda$ is a regular summability method. If $1/q \notin E'$, then the conclusion $s_n = o(1)$ is best possible.

If we take $\Lambda$ to be the Abel method, $q = (n)$ and $E = l_\infty$ in Corollary 1, then we have the answer to our question in the introduction regarding the best possible nature of $s_n = o(1)$ in the classical Tauberian case.

A later Tauberian theorem of Ingham's [2, Theorem 2] states that $\Sigma a_n = s (l)$ and $na_n = O(1)$ imply $s_n \to s$, where $(l)$ denotes Ingham summability. Taking $a_n = \mu(n)/n$, where $\mu(n)$ is the Möbius function, gives the prime number theorem. Although $(l)$ is not regular, for $E = l_r (0 < r \leq \infty)$ or $c_0$, we do have

**Corollary 2.** Suppose $\Sigma a_n = 0(l)$ and $qa \in E$ imply $s_n = o(1)$. If $1/q \notin E'$, then the conclusion $s_n = o(1)$ is best possible.

The proof follows immediately, using the following lemma, since in (i)–(iv) above each constructed $(a_k) \in l_1$.
Lemma 1. $\sum |a_k| < \infty$ implies $\Sigma a_k$ is summable (I).

Proof.

$$I_n = \frac{1}{n} \sum_{k=1}^{n} ka_k \left( \frac{n}{k} \right) = \sum_{k=1}^{n} a_k + O \left( \frac{1}{n} \sum_{k=1}^{n} k|a_k| \right).$$

But $\sum |a_k| < \infty$ implies $k|a_k| \to 0(C, 1)$. Hence $I_n = s_n + o(1)$.

3. Best possibility when $1/q \in E'$. If $1/q \in E'$, then $a \in S(q, E)$ does not imply that $s_n = o(1)$ is best possible. It is interesting to consider what is best possible in this case. As an illustration we have, when $E = c_0$, the following

**Theorem.** Suppose $1/q \in l_1$. Write $R(n) = \sum_{n+1}^{\infty} 1/q_k$. Then:

(a) $a \in S(q, c_0)$ implies $s_n = o(R(n))$.

(b) For all $p \not\in l_\infty$, $3a \in S(q, c_0)$ such that $p_n s_n / R(n) \nrightarrow 0$.

Proof. (a) was noted in (iv) above.

(b) Write $M(m + 1) = \sum_{m+1}^{\infty} 1/q_k$. Choose $n_1 > 1$ so that $p(n_1) > 1$. Define $a_k = R(n_1) / M(0) / p(n_1) q_k (1 \leq k \leq n_1)$. For $r \geq 1$ choose $n_{r+1} > n_r$ so that $R(n_r) / M(n_r, n_{r+1}) < 2$ and $p(n_{r+1}) > 2p(n_r)$. Define

$$a_k = \frac{(R(n_{r+1}) / p(n_{r+1}) - R(n_r) / p(n_r)) M(n_r, n_{r+1}) q_k}{R(n_r) / p(n_r)}.$$

Then, for $n_r < k \leq n_{r+1}$, $q_k a_k \leq 2/p(n_r) \to 0$, while $s_n$ decreases ($n > n_1$) and $s(n_r) = R(n_r) / p(n_r)$, completing the proof.

Finally, we have

**Corollary 3.** Suppose $1/q \in l_1$.

(a) If $qa \in c_0$ and $\Sigma a_n = 0(1)$, then $s_n = o(R(n))$.

(b) For all $p \not\in l_\infty$, $3a$ such that $qa \in c_0$, $\Sigma a_n = 0(1)$ but $p_n s_n / R(n) \nrightarrow 0$.

This corollary is an immediate consequence of the above Theorem since one can easily establish

**Lemma 2.** If $1/q \in l_1$, then $qa \in c_0$ implies $s_n - l_n \to 0$.

REFERENCES


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