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# A LEMMA OF ELIE CARTAN

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ABSTRACT. We provide in this paper an alternative proof of the lemma of E. Cartan which states that parallel translation of curvature and torsion locally determines an affine connection. Our proof uses covariant differentiation of tensor fields over mappings in place of Cartan's exterior differential calculus.

For background and explanation of the method to be used in this paper, see [1], [2], and [3]. Our notation is taken from [3].

To state the lemma, we need the following preliminaries.

Let $M$ be a smooth manifold with an affine connection, and let $U$ be a normal neighborhood of a point $p$ in $M$. Define the map $F$ with values in $U$ by $F(\vec{a}, t) = \exp_p(t\vec{a})$, for $\vec{a} = (a^1, \ldots, a^n)$ in $\mathbf{R}^n$ and $t$ sufficiently small. Now define $E(\vec{a}, t) = (e_1(\vec{a}, t), \ldots, e_n(\vec{a}, t))$, a field of bases parametrized by $F$, as follows. Choose a basis $E = (e_1, \ldots, e_n)$ for $M_p$, and then parallel translate $E$ out along geodesics $g_{\vec{a}}(t) = \exp_p(t\vec{a})$ to define $E(\vec{a}, t)$. Let $R^l_{ijk}(\vec{a}, t)$ and $T^i_{jk}(\vec{a}, t)$ denote the components of the curvature and torsion tensors with respect to $E(\vec{a}, t)$.

**Lemma (E. Cartan).** *The functions $R^l_{ijk}(\vec{a}, t)$ and $T^i_{jk}(\vec{a}, t)$ determine the connection on $U$.*

**Proof.** We have vector fields $\partial/\partial t, \partial/\partial a^1, \ldots, \partial/\partial a^n$ defined on the domain of $F$ in $(\vec{a}, t)$ space. Now set $Y_1 = F_*\partial/\partial a^1, \ldots, Y_n = F_*\partial/\partial a^n$; $Y_{n+1} = F_*\partial/\partial t$; these are vector fields over $F$ (see references). By the generalized structural equations [3, Proposition 2.2], we have

$$R_{Y_iY_{n+1}}Y_{n+1} = -\nabla_{\partial/\partial a^i}\nabla_{\partial/\partial t}Y_{n+1} + \nabla_{\partial/\partial t}\nabla_{\partial/\partial a^i}Y_{n+1} + \nabla_{[\partial/\partial t, \partial/\partial a^i]}Y_{n+1}.$$

But the first term on the right side of the last equation vanishes since for fixed $\vec{a}$, $t$ parametrizes the geodesic $g_{\vec{a}}(t)$. Also

$$\nabla_{\partial/\partial a^i}Y_{n+1} = \nabla_{\partial/\partial t}Y_i + T(Y_i, Y_{n+1}),$$

where $T$ is the torsion tensor. Thus we obtain

$$R_{Y_iY_{n+1}}Y_{n+1} = \nabla^2_{\partial/\partial t}Y_i + \nabla_{\partial/\partial t}T(Y_i, Y_{n+1}),$$

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the Jacobi equation for fixed $\vec{a}$. (We have used $[\partial/\partial t, \partial/\partial a^i] = 0$ here.)

Now $Y_i$ satisfies the initial conditions $Y_i(0) = 0$ and

$$(\nabla_{d/dt} Y_i)|_{t=0} = (\nabla_{d/dt} F_*(\partial/\partial a^i))(0) = (\nabla_{\partial/\partial a^i} F_*(d/dt))(0)$$

$$= \frac{\partial}{\partial a^i} g'_{\vec{a}}(0) = \frac{\partial}{\partial a^i} \sum a^j e_j = e_i.$$

Therefore if we set $Y_i = F_* \partial/\partial a^i = A_i^j e_j$, then $A_i^j$ are well-defined functions on $\mathbf{R}^{n+1}$, by existence and uniqueness of solutions to the Jacobi differential equation.

Next we look at

$$R_{Y_{n+1} Y_k} e_j = -\nabla_{\partial/\partial t}(\nabla_{\partial/\partial a^k} e_j) + \nabla_{\partial/\partial a^k} \nabla_{\partial/\partial t} e_j.$$

The last term on the right side of this equation vanishes, since $e_j$ are parallel along $g_{\vec{a}}(t)$. Setting $\Gamma^l_{kj} e_l = \nabla_{\partial/\partial a^k} e_j$, we have the system of ordinary differential equations

$$((d/dt)\Gamma^l_{kj}) e_l = R_{a^i e_i A^m_k e_m} e_j = a^i A^m_k R^l_{imj} e_l,$$

where $\vec{a}$ is kept fixed and we use the relation $Y_{n+1} = g'_{\vec{a}}(t) = a^i e_i$. Thus we have the system

$$d\Gamma^l_{kj}/dt = a^i A^m_k R^l_{mij}.$$

The initial conditions are given by $\Gamma^l_{kj} e_l|_{t=0} = \nabla_{\partial/\partial a^k} e_j|_0 = 0$ since $e_j$ are independent of $a^i$ at $t = 0$. Therefore the initial conditions are $\Gamma^l_{kj}|_{t=0} = 0$, and the $\Gamma^l_{kj}$ are uniquely determined by the data But these functions determine the connection in $U$ except at $p$, where it is uniquely determined by continuity.

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