AXIOMATIC SHAPE THEORY

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ABSTRACT. The notion of shape theory is so defined that, if \( H \) is a category and \( W \) is a subcategory of \( H \), all shape theories on \((H, W)\) are isomorphic and, under a mild condition, a shape theory on \((H, W)\) always exists. Additional theorems facilitate the comparison of shape theories constructed by various means.

1. The recent proliferation of shape theories ([1], [3], [5], [7], [8], [9], [11]) suggests the desirability of a unified treatment of their common features. This note is primarily a study of aspects of shape theory that can be dealt with in terms of category theory; many of our definitions and theorems contain no topology.

If \( A \) is a category, \( X \in A \) will mean that \( X \) is an object of \( A \) and \( f \in A(X, Y) \) will mean that \( f \) is a morphism of \( A \) with domain \( X \) and codomain \( Y \). If \( X \in A \), \( AX \) will denote the identity morphism in \( A(X, X) \). If \( B \) is a subcategory of \( A \) and \( X \in A \), let \( A(X, B) \) denote the class of morphisms of \( A \) with domain \( X \) and codomain an object of \( B \): \( A(X, B) = \bigcup_{Q \in B} A(X, Q) \). Throughout §§2—4, let \( H \) be a category and \( W \) a subcategory of \( H \). For example, \( H \) could be the category of topological spaces and homotopy classes and \( W \) could be the category of polyhedra and homotopy classes.

2. This section deals with the definition, existence and uniqueness of shape theories. Suppose \( G \) is a category and \( T: H \to G \) is a functor. A function \( v: H(X, W) \to G(D, G) \) is said to be linked by \( T \) if the following conditions hold:

\[ (L1) \] If \( Q \in W \) and \( k \in H(X, Q) \), then \( vk \in G(D, TQ) \).

\[ (L2) \] If \( r \in W(Q, P) \) and \( k \in H(X, Q) \), then \( v(rk) = (Tr)(vk) \).

(If \( H(X, W) \) and \( G(D, G) \) are endowed with the structure of a comma category [4, p. 28], then \( v \) is a functor.) If \( X \in H \), we say \( T \) is \( W \)-continuous at \( X \) if, given any \( D \in G \) and any function \( v: H(X, W) \to G(D, G) \) linked by \( T \), there is a unique \( g \in G(D, TX) \) such that \((TK)g = vk \) whenever \( k \in H(X, W) \). \( T \) is \( W \)-continuous if it is \( W \)-continuous at each object \( X \) of \( H \).

A pair \((C, C)\) is a shape theory on \((H, W)\) if \( C \) is a category and \( C: H \to C \) is a functor satisfying the following conditions:

\[ (S1) \] The objects of \( C \) are the objects of \( H \); if \( X \in H \), \( CX = X \).

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(S2) If \( Q \in W \) and \( s \in C(\lambda, Q) \), then there is just one \( k \in H(\lambda, \lambda) \) such that \( Ck = s \).

(S3) \( C \) is \( W \)-continuous.

A similar definition has been given by Holsztyński [5].

Suppose \((H, W)\) has the property that, if \( X \in H \), there is a set \( J \) of objects of \( W \) such that any morphism from \( X \) to an object of \( W \) factors through an object in \( J \). Then there is a shape theory on \((H, W)\). To see this, define a category \( S \) and a functor \( S: H \to S \) by the method of Holsztyński [5, p. 161, §4] (see also [7] and [8]). The objects of \( S \) are the objects of \( H \); if \( X \in H \), \( SX = X \). \( S(X, Y) \) is the class of functions \( v: H(Y, W) \to H(X, H) \) linked by the identity functor on \( H \). (The hypothesis on \((H, W)\) insures that the class \( S(X, Y) \) is a set.) If \( v \in S(X, Y) \) and \( u \in S(Y, Z) \), the composition \( uv \in S(X, X) \) is defined by \( (uv)k = v(uk) \) whenever \( k \in H(Z, W) \). If \( f \in H(X, Y) \), \( (Sf)k = kf \) whenever \( k \in H(Y, W) \).

Theorem 2.1. \((S, S)\) is a shape theory on \((H, W)\).

Proof. To verify (S2) suppose \( Q \in W \) and \( b \in S(Z, Q) \). If \( r \in H(Q, W) \), then \( (S(b(\lambda Q)))r = r(b(\lambda Q)) = br \). Hence

\[
S(b(\lambda Q)) = b.
\]

To see that \( b(\lambda Q) \) is unique, suppose \( c \in H(X, \lambda Q) \) and \( Sc = b \). Then \( c = (\lambda Q)c = (Sc)(\lambda Q) = b(\lambda Q) \).

To verify (S3), suppose \( v: H(X, W) \to S(Z, S) \) is a function linked by \( S \). If \( k \in H(X, W) \), by (S2) there is a unique \( gk \in H(Z, W) \) such that \( S(gk) = vk \). This defines a function \( g: H(X, W) \to H(X, H) \) that satisfies (L1). Suppose \( r \in W(Q, P) \) and \( k \in H(X, \lambda Q) \). Then

\[
S(g(rk)) = v(rk) = (Sr)(vk) = (Sr)(S(gk)) = S(r(gk)).
\]

By (S2), \( g(rk) = r(gk) \). Thus \( g \) satisfies (L2) as well as (L1), that is, \( g \in S(Z, X) \).

Next we show that \((Sk)g = vk \) whenever \( k \in H(X, W) \). Suppose \( Q \in W \) and \( k \in H(X, \lambda Q) \). If \( P \in W \) and \( r \in H(\lambda Q, P) \), then

\[
S(((Sk)g)r) = S(g(((Sk)r)r)) = S((Sk)g) = S((Sk)r) = S(g(rk)) = v(rk) = (Sr)(vk) = (Sr)(S((vk)(\lambda Q))) = S(r((vk)(\lambda Q))) = S((vk)r). \]

By (S2), \(((Sk)g)r = ((Sk)r) \). Hence \((Sk)g = vk \).
Suppose \( f \in S(Z, X) \) and \( (Sk)f = \nu k \) whenever \( k \in H(X, W) \). If \( r \in W(Q, P) \) and \( k \in H(X, Q) \), then
\[
(S(fk)r) = r(fk) = f((Sk)r) = ((Sk)f)r.
\]
So \( S(fk) = (Sk)f = \nu k = S(gk) \). By (S2), \( fk = gk \). Hence \( f = g \).

**Theorem 2.2.** Suppose \( C \) is a category and \( C : H \to C \) is a functor satisfying (S1) and (S2). If \( G \) is a category and \( T : H \to G \) is a \( W \)-continuous functor, then there is just one functor \( R : C \to G \) such that \( RC = T \).

**Proof.** If \( X \in C \), define \( RX = TX \). Suppose \( f \in C(X, Y) \). If \( Q \in W \) and \( k \in H(Y, Q) \), let \( u.k \in H(X, Q) \) be the unique morphism such that \( C(u.k) = (Ck)f \). This defines a function \( u : H(Y, W) \to H(X, H) \) satisfying (L1). If \( r \in W(g, P) \) and \( \gamma \in H(y, 2) \), then
\[
C(r(B/*)) = (CrKCOyl)) = (Cr)iCk)f = (Cu))/ = Cda/rA)).
\]
By (S2), \( r(uk) = u,(rk) \), that is, \( u \) satisfies (L2). Define \( v : H(Y, W) \to G(TX, G) \) by \( v.k = T(u.k) \). Since \( u \) is linked by the identity, \( v \) is linked by \( T \). Since \( T \) is \( W \)-continuous at \( T \), there is a unique \( Rf \in G(TX, TY) \) such that \( (Tk)(Rf) = v.k \) whenever \( k \in H(Y, W) \).

If \( Y = X \) and \( f = CX \), then \( k = u.k \) and \( (Tk)(G(RX)) = Tk = v.k = (Tk)(Rf) \) whenever \( k \in H(X, W) \). Since \( T \) is \( W \)-continuous at \( X \), \( G(RX) = Rf \).

Suppose \( f \in C(X, Y) \) and \( g \in C(Y, Z) \). If \( k \in H(Z, W) \), then
\[
C(u.g.k) = (Ck)gf = (Ck)(u.g)k = C(u.g.k).
\]
By (S2), \( u.g.k = u(u.g.k) \). Using this we compute
\[
(Tk)(R(g/f)) = v.g/k = T(u.g/k) = T(u.g.k) = v.g/k = T(v.g/k) = (Tk)(Rg)(Rf).
\]
Since \( T \) is \( W \)-continuous at \( Z \), \( R(g/f) = (Rg)(Rf) \). Thus \( R \) is a functor.

If \( f \in H(X, Y) \) and \( k \in H(Y, W) \), then \( C(u.cf.k) = (Ck)(Cf) = C(cf) \). By (S2), \( u.cf.k = kf \). Hence
\[
(Tk)(R(cf)) = v.cf/k = T(u.cf/k) = T(cf) = (Tk)(Tf).
\]
Since \( T \) is \( W \)-continuous at \( Y \), \( R(cf) = Tf \). Thus \( RC = T \).

Suppose \( R' : C \to G \) is a functor such that \( R'C = T \). If \( f \in C(X, Y) \) and \( k \in H(Y, W) \), then
\[
(Tk)(R'/f) = ((R'C)k)(R'/f) = R'(Ck)f = R'(C(u.k)) = T(u.k) = v/k = (Tk)(Rf).
\]
Since \( T \) is \( W \)-continuous at \( Y \), \( R'/f = Rf \). Thus \( R' = R \). This completes the proof of Theorem 2.2.

A functor \( R : C \to D \) is called an **isomorphism** if there is a functor \( R \) such that \( R' \) is an isomorphic functor to \( R \).
$P: D \to C$ such that $RP$ and $PR$ are identity functors. A fourfold application of Theorem 2.2 proves

**Theorem 2.3.** If each of $(C, C)$ and $(D, D)$ is a shape theory on $(\mathcal{H}, \mathcal{W})$, then there is a unique isomorphism $R: C \to D$ such that $RC = D$.

3. Throughout this section assume that $N$ is a subcategory of $\mathcal{W}$. We shall develop conditions under which a pair $(C, C)$ is a shape theory on both $(\mathcal{H}, \mathcal{W})$ and $(\mathcal{H}, \mathcal{N})$.

**Lemma 3.1.** If $T: \mathcal{H} \to \mathcal{G}$ is an $\mathcal{N}$-continuous functor, then $T$ is $\mathcal{W}$-continuous.

**Proof.** Suppose $\nu: \mathcal{H}(X, \mathcal{W}) \to \mathcal{G}(D, \mathcal{G})$ is linked by $T$. Let $u: \mathcal{H}(X, \mathcal{N}) \to \mathcal{G}(D, \mathcal{G})$ be the restriction of $\nu$; $u$ is linked by $T$. Since $T$ is $\mathcal{N}$-continuous at $X$, there is a $g \in \mathcal{G}(D, TX)$ such that, whenever $k \in \mathcal{H}(X, \mathcal{N})$, $(Tk)g = uk$.

Suppose $Q \in \mathcal{W}$ and $k \in \mathcal{H}(X, Q)$. If $r \in \mathcal{H}(Q, \mathcal{N})$, then

$$(Tr)(Tk)g = (T(rk))g = u(rk) = \nu(rk) = (Tr)(uk).$$

Since $T$ is $\mathcal{N}$-continuous at $Q$, $(Tk)g = uk$.

Suppose $f \in \mathcal{G}(D, TX)$ and $(Tk)f = uk$ whenever $k \in \mathcal{H}(X, \mathcal{W})$. Then, in particular, $(Tk)f = uk$ whenever $k \in \mathcal{H}(X, \mathcal{N})$. Since $T$ is $\mathcal{N}$-continuous at $X$, $f = g$. Thus $T$ is $\mathcal{W}$-continuous and Lemma 3.1 is proved.

An object $V$ of $\mathcal{H}$ is said to dominate an object $X$ of $\mathcal{H}$ if there exist morphisms $i \in \mathcal{H}(X, Y)$ and $j \in \mathcal{H}(Y, X)$ such that $ji = 1_X$.

**Lemma 3.2.** If $\mathcal{N}$ is full in $\mathcal{H}$, if every object of $\mathcal{W}$ is dominated by an object of $\mathcal{N}$, if $X \in \mathcal{H}$, and if $T: \mathcal{H} \to \mathcal{G}$ is a functor $\mathcal{W}$-continuous at $X$, then $T$ is $\mathcal{N}$-continuous at $X$.

**Proof.** Suppose $\nu: \mathcal{H}(X, \mathcal{N}) \to \mathcal{G}(D, \mathcal{G})$ is linked by $T$. If $Q \in \mathcal{W}$, let $M_Q \in \mathcal{N}$, $i_Q \in \mathcal{H}(Q, M_Q)$ and $j_Q \in \mathcal{H}(M_Q, Q)$ be such that $j_Q i_Q = HQ$. In particular, if $Q \in \mathcal{N}$, choose $M_Q = Q$ and $i_Q = j_Q = HQ$. If $k \in \mathcal{H}(X, Q)$, define $uk = (Tj_Q)(\nu(i_Qk))$. If $r \in \mathcal{W}(Q, \mathcal{P})$ and $k \in \mathcal{H}(X, Q)$, then

$$u(rk) = (Tj_P)(\nu(i_Prj_Qi_Qk)) = (Tj_P)(T(i_Prj_Q)(\nu(i_Qk))) = (Tr)(Tj_Q)(\nu(i_Qk)) = (Tr)(uk).$$

Thus the function $u: \mathcal{H}(X, \mathcal{W}) \to \mathcal{G}(D, \mathcal{G})$ is linked by $T$. Since $T$ is $\mathcal{W}$-continuous at $X$, there is a unique $g \in \mathcal{G}(D, TX)$ such that $(Tk)g = uk$ whenever $k \in \mathcal{H}(X, \mathcal{W})$. Since $u$ is an extension of $\nu$, $(Tk)g = uk$ whenever $k \in \mathcal{H}(X, \mathcal{N})$. Suppose $f \in \mathcal{G}(D, TX)$ and $(Tk)f = uk$ whenever $x \in \mathcal{H}(X, \mathcal{N})$. If $Q \in \mathcal{W}$ and $k \in \mathcal{H}(X, Q)$, then $(Tk)f = (Tj_Q)(T(i_Qk))f = (Tj_Q)(\nu(i_Qk)) = uk$. Since $T$ is $\mathcal{W}$-continuous at $X$, $f = g$. 

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Theorem 3.3. Suppose $\mathbf{N}$ is full in $\mathbf{W}$, each object of $\mathbf{W}$ is dominated by an object of $\mathbf{N}$, and $C: H \to \mathbf{C}$ is a functor. Then $(\mathbf{C}, \mathbf{C})$ is a shape theory on $(H, W)$ if and only if $(\mathbf{C}, \mathbf{C})$ is a shape theory on $(H, N)$.

Proof. Suppose $(\mathbf{C}, \mathbf{C})$ is a shape theory on $(H, N)$. By Lemma 3.1, $\mathbf{C}$ is $\mathbf{W}$-continuous. Suppose $Q \in \mathbf{W}$ and $s \in C(X, Q)$. Let $M \in \mathbf{N}$, $x \in H(M, Q)$ be such that $c = \mathbf{C}x$. There is a unique $f \in H(X, M)$ such that $Cf = (Cn)s$. Then $Cf(x) = (Cn)(Cf) = (Cn)(Cn) = s$. Suppose $g \in H(X, Q)$ is such that $Cg = s$. Then $Cg = (Cn)(Cg) = (Cn)s = Cn$. Since $(\mathbf{C}, \mathbf{C})$ satisfies (S2) as a shape theory on $(H, N)$, $ig = f$. Consequently $g = jg = jf$. Thus $(\mathbf{C}, \mathbf{C})$ satisfies (S2) as a shape theory on $(H, W)$.

By Lemma 3.2 any shape theory on $(H, W)$ is a shape theory on $(H, N)$.

4. In this section we suppose that $\mathbf{G}$ is a category, $T: H \to \mathbf{G}$ is a functor, $X \in H$, $A$ is a directed set, and $\{X_i \in H \mid i \in A\}$, $\{p_{ij} \in H(X_j, X_i) \mid i, j \in A, j > i\}$, $\{\rho_i \in H(X_i, X_j) \mid i \in A\}$ are such that the following conditions hold:

(B1) $\{\{X_i\}, \{p_{ij}\}\}$ is an inverse system.

(B2) $p_{ij}p_{ji} = p_i$ whenever $i, j \in A$ and $j > i$.

(B3) If $Q \in \mathbf{W}$ and $k \in H(X, Q)$, then there is an $i \in A$ and a $b \in H(X, Q)$ such that $k = bp_i$.

Theorem 4.1. If each $X_i$ is in $\mathbf{W}$, if each $p_{ij}$ is in $\mathbf{W}(X_j, X_i)$ and if $(TX, \{T\rho_{ij}\})$ is an inverse limit of $\{(TX_i), \{T\rho_{ij}\}\}$, then $T$ is $\mathbf{W}$-continuous at $X$.

Proof. Suppose $D \in \mathbf{G}$ and $v: H(X, W) \to G(D, \mathbf{G})$ is linked by $T$. For each $i \in A$, let $g_i = v\rho_i$. If $j > i$, $p_{ij}g_j = p_{ij}(v\rho_i) = v(p_{ij}\rho_i) = g_i$. By the definition of inverse limit, there is a unique $g \in G(D, TX)$ such that $(T\rho_{ij})g = g_i$ whenever $i \in A$. Suppose $Q \in \mathbf{W}$ and $k \in H(X, Q)$. By (B3) there is an $i \in A$ and a $b \in H(X, Q)$ such that $k = bp_i$. Hence

$$(Tk)g = (Tb)(T\rho_{ij})g = (Tb)(v\rho_i) = v(bp_i) = vk.$$ 

Suppose $f \in G(D, TX)$ and $(Tk)f = vk$ whenever $k \in H(X, W)$. Then, in particular, $(T\rho_{ij})f = g_i$ whenever $i \in A$. By the definition of inverse limit $f = g$.

Theorem 4.2. Suppose that, whenever $Q \in \mathbf{W}$, $i, j \in A$, $k_i \in H(X_i, Q)$ ($i = 1, 2$), and $k_{1p} = k_{2p}$, there is a $j > i$ such that $k_{1p_{ij}} = k_{2p_{ij}}$. If $T$ is $\mathbf{W}$-continuous, then $(TX, \{T\rho_{ij}\})$ is an inverse limit of $\{(TX_i), \{T\rho_{ij}\}\}$.

Proof. Suppose $\{g_i \in G(D, TX_i) \mid i \in A\}$ is such that $(T\rho_{ij})g_j = g_i$ whenever $i, j \in A$ and $j > i$. We must show that there is just one $g \in G(D, TX)$ such that $(T\rho_{ij})g = g_i$ whenever $i \in A$.

If $Q \in \mathbf{W}$ and $k \in H(X, Q)$, then, by (B3), there is an $m(k) \in A$ and a $b \in H(X_m(k), Q)$ such that $k = b_kp_{m(k)}$. Define $vk = (T\rho_{ij})g_{m(k)}$. Suppose $r \in W(Q, P)$ and $k \in H(X, Q)$. Let $i > m(k), m(k)$. Then
\[ rb_k p_{m(k)} p_{ij} = rb_k p_{m(k)} = rk = b_{rk} p_{m(rk)} = b_{rk} p_{m(rk)} p_{ij}. \]

By hypothesis there is a \( j > i \) such that \( rb_k p_{m(k)} p_{ij} = b_{rk} p_{m(rk)} p_{ij} \), which simplifies to \( rb_k p_{m(k)} p_{ij} = b_{rk} p_{m(rk)} p_{ij} \). Using this we compute

\[
(Tr)(vk) = (Tr)(Tb_k) g_{m(k)} = (Tr)(Tb_k) g_{m(k)} g_j = (Tb_k) g_{m(k)} = \nu(rk).
\]

Thus \( \nu: H(X, W) \to G(D, G) \) is linked by \( T \). Since \( T \) is \( W \)-continuous at \( X \), there is a unique \( g \) in \( G(D, TX) \) such that \( (Tk)g = \nu k \) whenever \( k \in H(X, W) \).

Suppose \( a \in A, Q \in W \) and \( k \in H(x_a, Q) \). Define \( e = k p_a \). Choose \( i > a, m(e) \). Then

\[ k p_a p_{ij} = b e p_{m(e)} = b e p_{m(e)} p_{ij}. \]

By hypothesis there is a \( j > i \) such that \( k p_a p_{ij} = b e p_{m(e)} p_{ij} \), which simplifies to \( k p_a p_{ij} = b e p_{m(e)} p_{ij} \). Using this we compute

\[
(Tk)(TP_{p_a}) g = (Te) g = (b e)(TP_{m(e)}) g_j = (Tk)(TP_{p_a}) g_j = (k) g_a.
\]

Since \( T \) is \( W \)-continuous at \( X_a \), \( (TP_{p_a}) g = g_a \).

Suppose \( f \in G(D, TX) \) and \( (TP_{p_a}) f = g_a \) whenever \( a \in A \). If \( k \in H(X, W) \), then

\[
(Tk)f = (Tb_k)(TP_{m(k)}) f = (Tb_k) g_{m(k)} = \nu k.
\]

Since \( T \) is \( W \)-continuous at \( X, f = g \).

5. This section contains an application of the theorems of §4. Let \( M \) be a full subcategory of the category of topological spaces and homotopy classes. If \( X, Y \in M \) and \( X \subset Y \), let \( i_YX \in M(X, Y) \) denote the homotopy class of the inclusion map and let \( \text{Nbd}(Y, X) \) denote the set of all \( N \in M \) such that \( X \subset \text{Int} N \) and \( N \subset Y \). A functor \( T \) from \( M \) to an arbitrary category \( G \) is called weakly continuous if, given \( X, Y \in M \) with \( X \) a closed subspace of \( Y \), \( (TX, \{Ti_{NX} | N \in \text{Nbd}(Y, X)\}) \) is an inverse limit of

\[
(\{TN | N \in \text{Nbd}(Y, X)\}, \{Ti_{LN} | L, N \in \text{Nbd}(Y, X), N \subset L\}).
\]

Let \( E \) be the full subcategory of \( M \) whose objects are those spaces in \( M \) that are neighborhood extensors for \( M \). We suppose that \( M \) satisfies the following conditions:

(A1) If \( X \in M \), then \( X \times I \in M \) (where \( I = [0, 1] \)).

(A2) If \( X, Y \in M \), if \( X \) is a closed subset of \( Y \), and if \( U \) is a neighborhood of \( X \), then there is a neighborhood \( N \) of \( X \) such that \( N \subset U \) and \( N \in M \).

(A3) If \( X \in M \), there is a \( Y \in M \) such that \( X \) is a closed subspace of \( Y \) and, given any neighborhood \( N \) of \( X \), there is a neighborhood \( N \) of \( X \) such that \( N \subset U \) and \( N \in E \).
Lemma 5.1. Suppose \( X, Y \in M \), \( X \) is a closed subspace of \( Y \), \( Q \in E \), \( L \in \text{Nbd}(Y, X) \), \( h_t \in M(L, Q) \) \((t = 0, 1)\) and \( h_0^i \text{LN} = h_1^i \text{LN} \). Then there is an \( N \) in \( \text{Nbd}(Y, X) \) such that \( N \subseteq L \) and \( h_0^i \text{LN} = h_1^i \text{LN} \).

Proof. Let \( f_i : L \rightarrow Q \) be a map in the homotopy class \( h_t \) \((t = 0, 1)\).
Since \( h_0^i \text{LN} = h_1^i \text{LN} \), there is a map \( m : X \times I \rightarrow Q \) such that \( m(x, t) = f_t x \) whenever \((x, t) \in X \times \{0, 1\}\). Let \( Z = X \times I \cup L \times \{0, 1\}; Z \) is a closed subspace of \( L \times I \). Define a map \( k : Z \rightarrow Q \) by \( k(x, t) = m(x, t) \) when \((x, t) \in X \times I; k(x, t) = f_t x \) when \((x, t) \in L \times \{0, 1\}\). Since \( Q \in E \), there is a neighborhood \( B \) of \( Z \) and an extension of \( k \) to a map \( j : B \rightarrow Q \). There is a neighborhood \( D \) of \( X \) such that \( D \times I \subseteq B \). By (A2) there is a neighborhood \( N \) of \( X \) such that \( N \subseteq D \) and \( N \in M \). The restriction of \( j \) to \( N \times I \) is a homotopy from \( f_0 | N \) to \( f_1 | N \). So \( h_0^i \text{LN} = h_1^i \text{LN} \).

Theorem 5.2. If \( G \) is a category and \( T : M \rightarrow G \) is a functor, then the following are equivalent:

1. \( T \) is weakly continuous.
2. \( T \) is \( E \)-continuous.

Proof that (1) implies (2). Suppose \( T \) is weakly continuous and \( X \in M \).
By (A3), \( X \) may be regarded as a closed subspace of a space \( Y \in M \) such that \( \text{Nbd}(Y, X) \) contains a cofinal subcollection \( \{X_i | i \in A\} \) of spaces in \( E \).
Since \( T \) is weakly continuous, \( TX \) is an inverse limit of \( \{TX_i | i \in A\} \). By (A2) and the definition of neighborhood extensor (B3) holds. By Theorem 4.1, \( T \) is \( E \)-continuous at \( X \).

Proof that (2) implies (1). Suppose \( T \) is \( E \)-continuous and \( X, Y \in M \) with \( X \) a closed subspace of \( Y \). By Lemma 5.1 and Theorem 4.2, \( TX \) is an inverse limit of \( \{TN | N \in \text{Nbd}(Y, X)\} \).

Theorem 5.2 shows that weak continuity can be used instead of \( E \)-continuity in our axioms for a shape theory on \((M, E)\). The category of metrizable spaces and homotopy classes furnishes an example of a category satisfying (A1)–(A3). (A3) holds because every metrizable space is embeddable as a closed set in a normed linear space \([10] \), every normed linear space is an extensor for metrizable spaces \([2, p. 188, Theorem 6.1] \), and open subsets of neighborhood extensors are neighborhood extensors \([6, p. 42, Proposition 6.1] \).

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