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AXIOMATIC SHAPE THEORY

PHILIP BACON

ABSTRACT. The notion of shape theory is so defined that, if $\mathbf{H}$ is a category and $\mathbf{W}$ is a subcategory of $\mathbf{H}$, all shape theories on $(\mathbf{H}, \mathbf{W})$ are isomorphic and, under a mild condition, a shape theory on $(\mathbf{H}, \mathbf{W})$ always exists. Additional theorems facilitate the comparison of shape theories constructed by various means.

1. The recent proliferation of shape theories ([1], [3], [5], [7], [8], [9], [11]) suggests the desirability of a unified treatment of their common features. This note is primarily a study of aspects of shape theory that can be dealt with in terms of category theory; many of our definitions and theorems contain no topology.

If $\mathbf{A}$ is a category, $X \in \mathbf{A}$ will mean that X is an object of $\mathbf{A}$ and $f \in \mathbf{A}(X, Y)$ will mean that f is a morphism of $\mathbf{A}$ with domain X and codomain Y. If $X \in \mathbf{A}$, $\mathbf{A}X$ will denote the identity morphism in $\mathbf{A}(X, X)$. If $\mathbf{B}$ is a subcategory of $\mathbf{A}$ and $X \in \mathbf{A}$, let $\mathbf{A}(X, \mathbf{B})$ denote the class of morphisms of $\mathbf{A}$ with domain X and codomain an object of $\mathbf{B}$: $\mathbf{A}(X, \mathbf{B}) = \bigcup\{\mathbf{A}(X, Q) | Q \in \mathbf{B}\}$. Throughout §§2–4, let $\mathbf{H}$ be a category and $\mathbf{W}$ a subcategory of $\mathbf{H}$. For example, $\mathbf{H}$ could be the category of topological spaces and homotopy classes and $\mathbf{W}$ could be the category of polyhedra and homotopy classes.

2. This section deals with the definition, existence and uniqueness of shape theories. Suppose $\mathbf{G}$ is a category and $T: \mathbf{H} \to \mathbf{G}$ is a functor. A function $v: \mathbf{H}(X, \mathbf{W}) \to \mathbf{G}(D, \mathbf{G})$ is said to be *linked by* T if the following conditions hold:

(L1) If $Q \in \mathbf{W}$ and $k \in \mathbf{H}(X, Q)$, then $vk \in \mathbf{G}(D, TQ)$.

(L2) If $r \in \mathbf{W}(Q, P)$ and $k \in \mathbf{H}(X, Q)$, then $v(rk) = (Tr)(vk)$.

(If $\mathbf{H}(X, \mathbf{W})$ and $\mathbf{G}(D, \mathbf{G})$ are endowed with the structure of a comma category [4, p. 28], then v is a functor.) If $X \in \mathbf{H}$, we say T is $\mathbf{W}$-*continuous at* X if, given any $D \in \mathbf{G}$ and any function $v: \mathbf{H}(X, \mathbf{W}) \to \mathbf{G}(D, \mathbf{G})$ linked by T, there is a unique $g \in \mathbf{G}(D, TX)$ such that $(TK)g = vk$ whenever $k \in \mathbf{H}(X, \mathbf{W})$. T is $\mathbf{W}$-*continuous* if it is $\mathbf{W}$-continuous at each object X of $\mathbf{H}$.

A pair $(\mathbf{C}, C)$ is a *shape theory on* $(\mathbf{H}, \mathbf{W})$ if $\mathbf{C}$ is a category and C: $\mathbf{H} \to \mathbf{C}$ is a functor satisfying the following conditions:

(S1) The objects of $\mathbf{C}$ are the objects of $\mathbf{H}$; if $X \in \mathbf{H}$, $CX = X$.

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(S2) If $Q \in \mathbf{W}$ and $s \in \mathbf{C}(X, Q)$, then there is just one $k \in \mathbf{H}(X, Q)$ such that $Ck = s$.

(S3) C is $\mathbf{W}$-continuous.

A similar definition has been given by Holsztyński [5].

Suppose $(\mathbf{H}, \mathbf{W})$ has the property that, if $X \in \mathbf{H}$, there is a *set* J of objects of $\mathbf{W}$ such that any morphism from X to an object of $\mathbf{W}$ factors through an object in J. Then there is a shape theory on $(\mathbf{H}, \mathbf{W})$. To see this, define a category $\mathbf{S}$ and a functor $S\colon \mathbf{H} \to \mathbf{S}$ by the method of Holsztyński [5, p. 161, § 4] (see also [7] and [8]). The objects of $\mathbf{S}$ are the objects of $\mathbf{H}$: if $X \in \mathbf{H}$, $SX = X$. $\mathbf{S}(X, Y)$ is the class of functions $v\colon \mathbf{H}(Y, \mathbf{W}) \to \mathbf{H}(X, \mathbf{H})$ linked by the identity functor on $\mathbf{H}$. (The hypothesis on $(\mathbf{H}, \mathbf{W})$ insures that the class $\mathbf{S}(X, Y)$ is a set.) If $v \in \mathbf{S}(X, Y)$ and $u \in \mathbf{S}(Y, Z)$, the composition $uv \in \mathbf{S}(X, X)$ is defined by $(uv)k = v(uk)$ whenever $k \in \mathbf{H}(Z, \mathbf{W})$. If $f \in \mathbf{H}(X, Y)$, $(Sf)k = kf$ whenever $k \in \mathbf{H}(Y, \mathbf{W})$.

Theorem 2.1. *$(\mathbf{S}, S)$ is a shape theory on $(\mathbf{H}, \mathbf{W})$.*

Proof. To verify (S2) suppose $Q \in \mathbf{W}$ and $b \in \mathbf{S}(Z, Q)$. If $r \in \mathbf{H}(Q, \mathbf{W})$, then $(S(b(\mathbf{H}Q)))r = r(b(\mathbf{H}Q)) = br$. Hence

$$S(b(\mathbf{H}Q)) = b. \tag{$*$}$$

To see that $b(\mathbf{H}Q)$ is unique, suppose $c \in \mathbf{H}(X, Q)$ and $Sc = b$. Then $c = (\mathbf{H}Q)c = (Sc)(\mathbf{H}Q) = b(\mathbf{H}Q)$.

To verify (S3), suppose $v\colon \mathbf{H}(X, \mathbf{W}) \to \mathbf{S}(Z, \mathbf{S})$ is a function linked by S. If $k \in \mathbf{H}(X, \mathbf{W})$, by (S2) there is a unique $gk \in \mathbf{H}(Z, \mathbf{W})$ such that $S(gk) = vk$. This defines a function $g\colon \mathbf{H}(X, \mathbf{W}) \to \mathbf{H}(X, \mathbf{H})$ that satisfies (L1). Suppose $r \in \mathbf{W}(Q, P)$ and $k \in \mathbf{H}(X, Q)$. Then

$$S(g(rk)) = v(rk) = (Sr)(vk) = (Sr)(S(gk)) = S(r(gk)).$$

By (S2), $g(rk) = r(gk)$. Thus g satisfies (L2) as well as (L1), that is, $g \in \mathbf{S}(Z, X)$.

Next we show that $(Sk)g = vk$ whenever $k \in \mathbf{H}(X, \mathbf{W})$. Suppose $Q \in \mathbf{W}$ and $k \in \mathbf{H}(X, Q)$. If $P \in \mathbf{W}$ and $r \in \mathbf{H}(Q, P)$, then

$$\begin{aligned}
S(((Sk)g)r) &= S(g((Sk)r)) && \text{by the definition of composition in } \mathbf{S},\\
&= S(g(rk)) && \text{by the definition of } S,\\
&= v(rk) && \text{by the definition of } g,\\
&= (Sr)(vk) && \text{since } v \text{ is linked by } S,\\
&= (Sr)(S((vk)(\mathbf{H}Q))) && \text{by } (*),\\
&= S(r((vk)(\mathbf{H}Q))) && \text{since } S \text{ is a functor},\\
&= S((vk)(r(\mathbf{H}Q))) && \text{since } vk \text{ is linked by the identity on } \mathbf{H},\\
&= S((vk)r).
\end{aligned}$$

By (S2), $((Sk)g)r = (vk)r$. Hence $(Sk)g = vk$.

Suppose $f \in \mathbf{S}(Z, X)$ and $(Sk)f = vk$ whenever $k \in \mathbf{H}(X, \mathbf{W})$. If $r \in \mathbf{W}(Q, P)$ and $k \in \mathbf{H}(X, Q)$, then

$$(S(fk))r = r(fk) = f(rk) = f((Sk)r) = ((Sk)f)r.$$

So $S(fk) = (Sk)f = vk = S(gk)$. By (S2), $fk = gk$. Hence $f = g$.

Theorem 2.2. *Suppose* $\mathbf{C}$ *is a category and* $C: \mathbf{H} \to \mathbf{C}$ *is a functor satisfying* (S1) *and* (S2). *If* $\mathbf{G}$ *is a category and* $T: \mathbf{H} \to \mathbf{G}$ *is a* $\mathbf{W}$*-continuous functor, then there is just one functor* $R: \mathbf{C} \to \mathbf{G}$ *such that* $RC = T$.

Proof. If $X \in \mathbf{C}$, define $RX = TX$. Suppose $f \in \mathbf{C}(X, Y)$. If $Q \in \mathbf{W}$ and $k \in \mathbf{H}(Y, Q)$, let $u_f k \in \mathbf{H}(X, Q)$ be the unique morphism such that $C(u_f k) = (Ck)f$. This defines a function $u_f: \mathbf{H}(Y, \mathbf{W}) \to \mathbf{H}(X, \mathbf{H})$ satisfying (L1). If $r \in \mathbf{W}(Q, P)$ and $k \in \mathbf{H}(Y, Q)$, then

$$C(r(u_f k)) = (Cr)(C(u_f k)) = (Cr)(Ck)f = (C(rk))f = C(u_f(rk)).$$

By (S2), $r(u_f k) = u_f(rk)$, that is, u_f satisfies (L2). Define $v_f: \mathbf{H}(Y, \mathbf{W}) \to \mathbf{G}(TX, \mathbf{G})$ by $v_f k = T(u_f k)$. Since u_f is linked by the identity, v_f is linked by T. Since T is $\mathbf{W}$-continuous at T, there is a unique $Rf \in \mathbf{G}(TX, TY)$ such that $(Tk)(Rf) = v_f k$ whenever $k \in \mathbf{H}(Y, \mathbf{W})$.

If $Y = X$ and $f = \mathbf{C}X$, then $k = u_f k$ and $(Tk)(\mathbf{G}(RX)) = Tk = v_f k = (Tk)(Rf)$ whenever $k \in \mathbf{H}(X, \mathbf{W})$. Since T is $\mathbf{W}$-continuous at X, $\mathbf{G}(RX) = Rf$.

Suppose $f \in \mathbf{C}(X, Y)$ and $g \in \mathbf{C}(Y, Z)$. If $k \in \mathbf{H}(Z, \mathbf{W})$, then

$$C(u_{gf} k) = (Ck)gf = (C(u_g k))f = C(u_f(u_g k)).$$

By (S2), $u_{gf} k = u_f(u_g k)$. Using this we compute

$$\begin{aligned}(Tk)(R(gf)) &= v_{gf} k = T(u_{gf} k) = T(u_f(u_g k)) = v_f(u_g k) \\ &= (T(u_g k))(Rf) = (v_g k)(Rf) = (Tk)(Rg)(Rf).\end{aligned}$$

Since T is $\mathbf{W}$-continuous at Z, $R(gf) = (Rg)(Rf)$. Thus R is a functor.

If $f \in \mathbf{H}(X, Y)$ and $k \in \mathbf{H}(Y, \mathbf{W})$, then $C(u_{Cf} k) = (Ck)(Cf) = C(kf)$. By (S2), $u_{Cf} k = kf$. Hence

$$(Tk)(R(Cf)) = v_{Cf} k = T(u_{Cf} k) = T(kf) = (Tk)(Tf).$$

Since T is $\mathbf{W}$-continuous at Y, $R(Cf) = Tf$. Thus $RC = T$.

Suppose $R': \mathbf{C} \to \mathbf{G}$ is a functor such that $R'C = T$. If $f \in \mathbf{C}(X, Y)$ and $k \in \mathbf{H}(Y, \mathbf{W})$, then

$$(Tk)(R'f) = ((R'C)k)(R'f) = R'((Ck)f) = R'(C(u_f k)) = T(u_f k) = v_f k = (Tk)(Rf).$$

Since T is $\mathbf{W}$-continuous at Y, $R'f = Rf$. Thus $R' = R$. This completes the proof of Theorem 2.2.

A functor $R: \mathbf{C} \to \mathbf{D}$ is called an *isomorphism* if there is a functor

$P: \mathbf{D} \to \mathbf{C}$ such that RP and PR are identity functors. A fourfold application of Theorem 2.2 proves

Theorem 2.3. *If each of* $(\mathbf{C}, C)$ *and* $(\mathbf{D}, D)$ *is a shape theory on* $(\mathbf{H}, \mathbf{W})$, *then there is a unique isomorphism* $R: \mathbf{C} \to \mathbf{D}$ *such that* $RC = D$.

3. Throughout this section assume that $\mathbf{N}$ is a subcategory of $\mathbf{W}$. We shall develop conditions under which a pair $(\mathbf{C}, C)$ is a shape theory on both $(\mathbf{H}, \mathbf{W})$ and $(\mathbf{H}, \mathbf{N})$.

Lemma 3.1. *If* $T: \mathbf{H} \to \mathbf{G}$ *is an* $\mathbf{N}$*-continuous functor, then* T *is* $\mathbf{W}$*-continuous.*

Proof. Suppose $v: \mathbf{H}(X, \mathbf{W}) \to \mathbf{G}(D, \mathbf{G})$ is linked by T. Let $u: \mathbf{H}(X, \mathbf{N}) \to \mathbf{G}(D, \mathbf{G})$ be the restriction of v; u is linked by T. Since T is $\mathbf{N}$-continuous at X, there is a $g \in \mathbf{G}(D, TX)$ such that, whenever $k \in \mathbf{H}(X, \mathbf{N})$, $(Tk)g = uk$.

Suppose $Q \in \mathbf{W}$ and $k \in \mathbf{H}(X, Q)$. If $r \in \mathbf{H}(Q, \mathbf{N})$, then

$$(Tr)(Tk)g = (T(rk))g = u(rk) = v(rk) = (Tr)(vk).$$

Since T is $\mathbf{N}$-continuous at Q, $(Tk)g = vk$.

Suppose $f \in \mathbf{G}(D, TX)$ and $(Tk)f = vk$ whenever $k \in \mathbf{H}(X, \mathbf{W})$. Then, in particular, $(Tk)f = uk$ whenever $k \in \mathbf{H}(X, \mathbf{N})$. Since T is $\mathbf{N}$-continuous at X, $f = g$. Thus T is $\mathbf{W}$-continuous and Lemma 3.1 is proved.

An object Y of $\mathbf{H}$ is said to *dominate* an object X of $\mathbf{H}$ if there exist morphisms $i \in \mathbf{H}(X, Y)$ and $j \in \mathbf{H}(Y, X)$ such that $ji = \mathbf{H}X$.

Lemma 3.2. *If* $\mathbf{N}$ *is full in* $\mathbf{H}$, *if every object of* $\mathbf{W}$ *is dominated by an object of* $\mathbf{N}$, *if* $X \in \mathbf{H}$, *and if* $T: \mathbf{H} \to \mathbf{G}$ *is a functor* $\mathbf{W}$*-continuous at* X, *then* T *is* $\mathbf{N}$*-continuous at* X.

Proof. Suppose $v: \mathbf{H}(X, \mathbf{N}) \to \mathbf{G}(D, \mathbf{G})$ is linked by T. If $Q \in \mathbf{W}$, let $M_Q \in \mathbf{N}$, $i_Q \in \mathbf{H}(Q, M_Q)$ and $j_Q \in \mathbf{H}(M_Q, Q)$ be such that $j_Q i_Q = \mathbf{H}Q$. In particular, if $Q \in \mathbf{N}$, choose $M_Q = Q$ and $i_Q = j_Q = \mathbf{H}Q$. If $k \in \mathbf{H}(X, Q)$, define $uk = (Tj_Q)(v(i_Q k))$. If $r \in \mathbf{W}(Q, P)$ and $k \in \mathbf{H}(X, Q)$, then

$$\begin{aligned} u(rk) &= (Tj_P)(v(i_P rk)) = (Tj_P)(v(i_P r j_Q i_Q k)) = (Tj_P)(T(i_P r j_Q))(v(i_Q k)) \\ &= (Tr)(Tj_Q)(v(i_Q k)) = (Tr)(uk). \end{aligned}$$

Thus the function $u: \mathbf{H}(X, \mathbf{W}) \to \mathbf{G}(D, \mathbf{G})$ is linked by T. Since T is $\mathbf{W}$-continuous at X, there is a unique $g \in \mathbf{G}(D, TX)$ such that $(Tk)g = uk$ whenever $k \in \mathbf{H}(X, \mathbf{W})$. Since u is an extension of v, $(Tk)g = vk$ whenever $k \in \mathbf{H}(X, \mathbf{N})$. Suppose $f \in \mathbf{G}(D, TX)$ and $(Tk)f = vk$ whenever $x \in \mathbf{H}(X, \mathbf{N})$. If $Q \in \mathbf{W}$ and $k \in \mathbf{H}(X, Q)$, then $(Tk)f = (Tj_Q)(T(i_Q k))f = (Tj_Q)(v(i_Q k)) = uk$. Since T is $\mathbf{W}$-continuous at X, $f = g$.

Theorem 3.3. *Suppose* $\mathbf{N}$ *is full in* $\mathbf{W}$, *each object of* $\mathbf{W}$ *is dominated by an object of* $\mathbf{N}$, *and* $C\colon \mathbf{H} \to \mathbf{C}$ *is a functor. Then* $(\mathbf{C}, C)$ *is a shape theory on* $(\mathbf{H}, \mathbf{W})$ *if and only if* $(\mathbf{C}, C)$ *is a shape theory on* $(\mathbf{H}, \mathbf{N})$.

Proof. Suppose $(\mathbf{C}, C)$ is a shape theory on $(\mathbf{H}, \mathbf{N})$. By Lemma 3.1, C is $\mathbf{W}$-continuous. Suppose $Q \in \mathbf{W}$ and $s \in \mathbf{C}(X, Q)$. Let $M \in \mathbf{N}$, $i \in \mathbf{H}(Q, M)$ and $j \in \mathbf{H}(M, Q)$ be such that $ji = 1Q$. There is a unique $f \in \mathbf{H}(X, M)$ such that $Cf = (Ci)s$. Then $C(jf) = (Cj)(Cf) = (Cj)(Ci)s = s$. Suppose $g \in \mathbf{H}(X, Q)$ is such that $Cg = s$. Then $C(ig) = (Ci)(Cg) = (Ci)s = Cf$. Since $(\mathbf{C}, C)$ satisfies (S2) as a shape theory on $(\mathbf{H}, \mathbf{N})$, $ig = f$. Consequently $g = jig = jf$. Thus $(\mathbf{C}, C)$ satisfies (S2) as a shape theory on $(\mathbf{H}, \mathbf{W})$.

By Lemma 3.2 any shape theory on $(\mathbf{H}, \mathbf{W})$ is a shape theory on $(\mathbf{H}, \mathbf{N})$.

4. In this section we suppose that $\mathbf{G}$ is a category, $T\colon \mathbf{H} \to \mathbf{G}$ is a functor, $X \in \mathbf{H}$, A is a directed set, and $\{X_i \in \mathbf{H} \mid i \in A\}$, $\{p_{ij} \in \mathbf{H}(X_j, X_i) \mid i, j \in A, j > i\}$, $\{p_i \in \mathbf{H}(X, X_i) \mid i \in A\}$ are such that the following conditions hold:

(B1) $(\{X_i\}, \{p_{ij}\})$ is an inverse system.

(B2) $p_{ij}p_j = p_i$ whenever $i, j \in A$ and $j > i$.

(B3) If $Q \in \mathbf{W}$ and $k \in \mathbf{H}(X, Q)$, then there is an $i \in A$ and a $b \in \mathbf{H}(X_i, Q)$ such that $k = bp_i$.

Theorem 4.1. *If each* X_i *is in* $\mathbf{W}$, *if each* p_{ij} *is in* $\mathbf{W}(X_j, X_i)$ *and if* $(TX, \{Tp_i\})$ *is an inverse limit of* $(\{TX_i\}, \{Tp_{ij}\})$, *then* T *is* $\mathbf{W}$-*continuous at* X.

Proof. Suppose $D \in \mathbf{G}$ and $v\colon \mathbf{H}(X, \mathbf{W}) \to \mathbf{G}(D, \mathbf{G})$ is linked by T. For each $i \in A$, let $g_i = vp_i$. If $j > i$, $p_{ij}g_j = p_{ij}(vp_i) = v(p_{ij}p_j) = vp_i = g_i$. By the definition of inverse limit, there is a unique $g \in \mathbf{G}(D, TX)$ such that $(Tp_i)g = g_i$ whenever $i \in A$. Suppose $Q \in \mathbf{W}$ and $k \in \mathbf{H}(X, Q)$. By (B3) there is an $i \in A$ and a $b \in \mathbf{H}(X_i, Q)$ such that $k = bp_i$. Hence

$$(Tk)g = (Tb)(Tp_i)g = (Tb)(vp_i) = v(bp_i) = vk.$$

Suppose $f \in \mathbf{G}(D, TX)$ and $(TK)f = vk$ whenever $k \in \mathbf{H}(X, \mathbf{W})$. Then, in particular, $(Tp_i)f = g_i$ whenever $i \in A$. By the definition of inverse limit $f = g$.

Theorem 4.2. *Suppose that, whenever* $Q \in \mathbf{W}$, $i \in A$, $k_t \in \mathbf{H}(X_i, Q)$ $(t = 1, 2)$, *and* $k_1p_i = k_2p_i$, *there is a* $j > i$ *such that* $k_1p_{ij} = k_2p_{ij}$. *If* T *is* $\mathbf{W}$-*continuous, then* $(TX, \{Tp_i\})$ *is an inverse limit of* $(\{TX_i\}, \{Tp_{ij}\})$.

Proof. Suppose $\{g_i \in \mathbf{G}(D, TX_i) \mid i \in A\}$ is such that $(Tp_{ij})g_j = g_i$ whenever $i, j \in A$ and $j > i$. We must show that there is just one $g \in \mathbf{G}(D, TX)$ such that $(Tp_i)g = g_i$ whenever $i \in A$.

If $Q \in \mathbf{W}$ and $k \in \mathbf{H}(X, Q)$, then, by (B3), there is an $m(k) \in A$ and a $b_k \in \mathbf{H}(X_{m(k)}, Q)$ such that $k = b_kp_{m(k)}$. Define $vk = (Tb_k)g_{m(k)}$. Suppose $r \in \mathbf{W}(Q, P)$ and $k \in \mathbf{H}(X, Q)$. Let $i > m(k), m(rk)$. Then

$$rb_k p_{m(k)i} p_i = rb_k p_{m(k)} = rk = b_{rk} p_{m(rk)} = b_{rk} p_{m(rk)i} p_i.$$

By hypothesis there is a $j > i$ such that $rb_k p_{m(k)i} p_{ij} = b_{rk} p_{m(rk)i} p_{ij}$, which simplifies to $rb_k p_{m(k)j} = b_{rk} p_{m(rk)j}$. Using this we compute

$$\begin{aligned}(Tr)(vk) &= (Tr)(Tb_k) g_{m(k)} = (Tr)(Tb_k)(Tp_{m(k)j}) g_j \\ &= (Tb_{rk})(Tp_{m(rk)j}) g_j = (Tb_{rk}) g_{m(rk)} = v(rk).\end{aligned}$$

Thus v: $\mathbf{H}(X, \mathbf{W}) \to \mathbf{G}(D, \mathbf{G})$ is linked by T. Since T is $\dot{\mathbf{W}}$-continuous at X, there is a unique g in $\mathbf{G}(D, TX)$ such that $(Tk)g = vk$ whenever $k \in \mathbf{H}(X, \mathbf{W})$.

Suppose $a \in A$, $Q \in \mathbf{W}$ and $k \in \mathbf{H}(X_a, Q)$. Define $e = kp_a$. Choose $i > a$, $m(e)$. Then

$$kp_{ai} p_i = kp_a = e = b_e p_{m(e)} = b_e p_{m(e)i} p_i.$$

By hypothesis there is a $j > i$ such that $kp_{ai} p_{ij} = b_e p_{m(e)i} p_{ij}$, which simplifies to $kp_{aj} = b_e p_{m(e)j}$. Using this we compute

$$(Tk)(Tp_a)g = (Te)g = ve = (Tb_e) g_{m(e)} = (b_e)(Tp_{m(e)j}) g_j = (Tk)(Tp_{aj}) g_j = (k) g_a.$$

Since T is $\mathbf{W}$-continuous at X_a, $(Tp_a)g = g_a$.

Suppose $f \in \mathbf{G}(D, TX)$ and $(Tp_a)f = g_a$ whenever $a \in A$. If $k \in \mathbf{H}(X, \mathbf{W})$, then

$$(Tk)f = (Tb_k)(Tp_{m(k)})f = (Tb_k) g_{m(k)} = vk.$$

Since T is $\mathbf{W}$-continuous at X, $f = g$.

5. This section contains an application of the theorems of §4. Let $\mathbf{M}$ be a full subcategory of the category of topological spaces and homotopy classes. If $X, Y \in \mathbf{M}$ and $X \subset Y$, let $i_{YX} \in \mathbf{M}(X, Y)$ denote the homotopy class of the inclusion map and let $\mathrm{Nbd}(Y, X)$ denote the set of all $N \in \mathbf{M}$ such that $X \subset \mathrm{Int}\, N$ and $N \subset Y$. A functor T from $\mathbf{M}$ to an arbitrary category $\mathbf{G}$ is called *weakly continuous* if, given $X, Y \in \mathbf{M}$ with X a closed subspace of Y, $(TX, \{Ti_{NX} \mid N \in \mathrm{Nbd}(Y, X)\})$ is an inverse limit of

$$(\{TN \mid N \in \mathrm{Nbd}(Y, X)\}, \{Ti_{LN} \mid L, N \in \mathrm{Nbd}(Y, X), N \subset L\}).$$

Let $\mathbf{E}$ be the full subcategory of $\mathbf{M}$ whose objects are those spaces in $\mathbf{M}$ that are neighborhood extensors for $\mathbf{M}$. We suppose that $\mathbf{M}$ satisifes the following conditions:

(A1) If $X \in \mathbf{M}$, then $X \times I \in \mathbf{M}$ (where $I = [0, 1]$).

(A2) If $X, Y \in \mathbf{M}$, if X is a closed subset of Y, and if U is a neighborhood of X, then there is a neighborhood N of X such that $N \subset U$ and $N \in \mathbf{M}$.

(A3) If $X \in \mathbf{M}$, there is a $Y \in \mathbf{M}$ such that X is a closed subspace of Y and, given any neighborhood U of X, there is a neighborhood N of X such that $N \subset U$ and $N \in \mathbf{E}$.

Lemma 5.1. *Suppose* $X, Y \in \mathbf{M}$, X *is a closed subspace of* Y, $Q \in \mathbf{E}$, $L \in \mathrm{Nbd}(Y, X)$, $h_t \in \mathbf{M}(L, Q)$ $(t = 0, 1)$ *and* $h_0 i_{LX} = h_1 i_{LX}$. *Then there is an* N *in* $\mathrm{Nbd}(Y, X)$ *such that* $N \subset L$ *and* $h_0 i_{LN} = h_1 i_{LN}$.

Proof. Let $f_t: L \to Q$ be a map in the homotopy class h_t $(t = 0, 1)$. Since $h_0 i_{LX} = h_1 i_{LX}$, there is a map $m: X \times I \to Q$ such that $m(x, t) = f_t x$ whenever $(x, t) \in X \times \{0, 1\}$. Let $Z = X \times I \cup L \times \{0, 1\}$; Z is a closed subspace of $L \times I$. Define a map $k: Z \to Q$ by $k(x, t) = m(x, t)$ when $(x, t) \in X \times I$; $k(x, t) = f_t x$ when $(x, t) \in L \times \{0, 1\}$. Since $Q \in \mathbf{E}$, there is a neighborhood B of Z and an extension of k to a map $j: B \to Q$. There is a neighborhood D of X such that $D \times I \subset B$. By (A2) there is a neighborhood N of X such that $N \subset D$ and $N \in \mathbf{M}$. The restriction of j to $N \times I$ is a homotopy from $f_0 | N$ to $f_1 | N$. So $h_0 i_{LN} = h_1 i_{LN}$.

Theorem 5.2. *If* $\mathbf{G}$ *is a category and* $T: \mathbf{M} \to \mathbf{G}$ *is a functor, then the following are equivalent*:

(1) T *is weakly continuous.*

(2) T *is* $\mathbf{E}$-*continuous.*

Proof that (1) implies (2). Suppose T is weakly continuous and $X \in \mathbf{M}$. By (A3), X may be regarded as a closed subspace of a space Y in $\mathbf{M}$ such that $\mathrm{Nbd}(Y, X)$ contains a cofinal subcollection $\{X_i | i \in A\}$ of spaces in $\mathbf{E}$. Since T is weakly continuous, TX is an inverse limit of $\{TX_i | i \in A\}$. By (A2) and the definition of neighborhood extensor (B3) holds. By Theorem 4.1, T is $\mathbf{E}$-continuous at X.

Proof that (2) implies (1). Suppose T is $\mathbf{E}$-continuous and $X, Y \in \mathbf{M}$ with X a closed subspace of Y. By Lemma 5.1 and Theorem 4.2, TX is an inverse limit of $\{TN | N \in \mathrm{Nbd}(Y, X)\}$.

Theorem 5.2 shows that weak continuity can be used instead of $\mathbf{E}$-continuity in our axioms for a shape theory on $(\mathbf{M}, \mathbf{E})$. The category of metrizable spaces and homotopy classes furnishes an example of a category satisfying (A1)–(A3). (A3) holds because every metrizable space is embeddable as a closed set in a normed linear space [10], every normed linear space is an extensor for metrizable spaces [2, p. 188, Theorem 6.1], and open subsets of neighborhood extensors are neighborhood extensors [6, p. 42, Proposition 6.1].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611