ALMOST DETERMINISTIC $\omega$-AUTOMATA WITH
EXISTENTIAL OUTPUT CONDITION

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ABSTRACT. The theorem on reduction in the nondeterminateness de-

gree of $\omega$-automata has been formulated.

1. Notation. $\omega$ denotes the set of natural numbers. An ordinal will be

identified with the set of all its predecessors. $|A|$ and $P(A)$ denote the

cardinality and the power set of $A$. For a set $A$ and an ordinal $\alpha$, denote

$A^\alpha$ to be the set of all $\alpha$-sequences over $A$, $A^\alpha = \{v|v: \alpha \rightarrow A\}$. $A^* = \bigcup_{n} \omega^n\alpha$.

We shall write $l(v) = n$ if $v \in A^n$. If $v_1 \in A^*, v_2 \in A^a (B \subseteq A^*, C \subseteq A^a)$ for an

ordinal, then $v_1v_2 (B,C)$ will denote the result of concatenating $v_1$ with $v_2$

($B$ with $C$).

For a function $f: A \rightarrow B$, define

$ln(f) = \{b|b \in B, |f^{-1}(b)| \geq n\}$.

2. $\omega$-definability. We shall keep the terminology of [6].

A (nondeterministic) automaton over an alphabet $\Sigma$ is a quadruple $\mathcal{A} =$

$(S, M, S_0, F)$ where $S$ is a finite set, the set of states, $M$ is a function $M: S \times$

$\Sigma \rightarrow P(S)$, the transition function, $S_0 \subseteq S$ is the set of initial states, and $F \subseteq S$

is the set of final states.

The rank of $\mathcal{A}$ is the least number $n$ such that $|S_0| \leq n$ and $|M(s, \sigma)| \leq$

$n$ for every $s \in S$, $\sigma \in \Sigma$. An automaton of rank 1 is called deterministic (d.).

An automaton $\mathcal{A} = (S, M, S_0, F)$ is called limitary deterministic (l. d.) if

there is a d. automaton $\mathcal{B} = (T, N, T_0, G)$ over $\Sigma$ with $G = F$ and $N \subseteq M$.

Given $n < \omega$. An $\omega$-run on $v \in \Sigma^n$ is a function $r: n + 1 \rightarrow S$ such that

$r(0) \in S_0$ and $r(i + 1) \in M(r(i), v(i))$, $i < n$.

An $\omega$-run on $v \in \Sigma^\omega$ is a function $r: \omega \rightarrow S$ satisfying the above for any

$i < \omega$.

A word $v \in \Sigma^n$, $n < \omega$, is accepted by $\mathcal{A}$ if there is an $\mathcal{A}$-run on $v$ such that

$r(n) \in F$. The set of all words $v \in \Sigma^*$ accepted by $\mathcal{A}$ will be denoted by

$L(\mathcal{A})$. A set $A \subseteq \Sigma^*$ is called regular if for some automaton $\mathcal{A}$, $L(\mathcal{A}) = A$.

Proceeding to $\omega$-sequences, we introduce two different output conditions

attached to two different notions of finite automata. And so with the automaton
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\( \mathbb{U} = (S, M, s_0, F) \) we associate, following J. R. Büchi [1], the existential rule of \( \omega \)-definability:

\( \mathbb{U} \) defines \( v \in \Sigma^\omega \) iff there is an \( \mathbb{U} \)-run on \( v \) such that \( In(r) \cap F \neq \emptyset \).

The set of all \( v \in \Sigma^\omega \) defined by \( \mathbb{U} \) will be denoted by \( E(\mathbb{U}) \). A set \( A \subseteq \Sigma^\omega \) is existentially definable (ED) if for some automaton \( \mathbb{U} \), \( E(\mathbb{U}) = A \).

A set \( A \subseteq \Sigma^\omega \) is ED of rank \( n \) if there is an automaton \( \mathbb{U} \) of rank \( n \) such that \( E(\mathbb{U}) = A \). An ED set of rank 1 will be called deterministic (d.).

The second notion of the finite acceptor is due to D. Muller [4].

A Muller automaton over \( \Sigma \) is a system \( \mathbb{U} = (S, M, s_0, F) \) where \( S \) is a finite set, \( M : S \times \Sigma \rightarrow S \), \( s_0 \in S \), and \( F \subseteq P(S) \) is the set of designated subsets of \( S \).

The Muller (universal) rule of \( \omega \)-definability states:

\( \mathbb{U} \) defines \( v \in \Sigma^\omega \) iff there is a function \( r : \omega \rightarrow S \) satisfying \( r(0) = s_0 \), \( r(i + 1) = M(r(i), \nu(i)), i < \omega \), and such that \( In(r) \in F \).

The set of all \( \omega \)-sequences defined in such a manner will be denoted by \( U(\mathbb{U}) \). A set \( A \subseteq \Sigma^\omega \) is called universally definable (UD) if for some Muller automaton \( \mathbb{U} \), \( U(\mathbb{U}) = A \).

By the fundamental result of McNaughton [3] we have

**Theorem 1.** Given a set \( A \subseteq \Sigma^\omega \) \( A \) is ED if and only if \( A \) is UD.

3. Rank and limitary determinism of ED sets. It is trivially verifiable that there are ED sets which are not of rank 1. This fact naturally raises the question of the possible reductions in the degree of nondeterminateness of such sets. In answer to this we have

**Theorem 2.** For every automaton \( \mathbb{U} \) there exists a l.d. automaton \( \mathbb{B} \) of rank 2 such that \( E(\mathbb{B}) = E(\mathbb{U}) \).

**Proof.** Let \( \mathbb{U} = (S, M, s_0, F) \) be a Muller automaton with \( F = \{A_j\}_{j < n} \).

Construct the set \( T = \{(a_0, \ldots , a_{n-1})s|a_j \in P(A_j) \text{ or } a_j = \rho, s \in S, \rho \notin P(S), \text{ and } \}

and

\( N((a_0, \ldots , a_{n-1})s, o) = \{(b_0, \ldots , b_{n-1})M(s, o)\} \)

and

\( b_j = a_j \cup \{s\} \) if \( s \in A_j \) and \( a_j \neq A_j, a_j \neq \rho \),

\( = \emptyset \) if \( s \in A_j \) and \( a_j = A_j, (a_j \neq \rho) \),

\( = \rho \) otherwise.

Let us define the automaton \( \mathbb{B} = (S \cup T, H, \{s_0\}, G) \) where \( H(s, o) = \{(\emptyset, \ldots , \emptyset) s', s'^*\} \) for \( s' = M(s, o) \), \( s \in S \), \( H|T \times \Sigma = N \) and \( G = \{(a_0, \ldots , a_{n-1})s|a_j = A_j \text{ for some } j \in n\} \). \( \mathbb{B} \) is l.d. and of rank 2.

With the above, for any function \( v \in \Sigma^\omega \) the following occurrences are equivalent:
(1) There is a function \( r \in (S \cup T)^\omega \) such that \( r(i + 1) \in H(r(i), v(i)) \), \( i < \omega \), and \( LN(r) \cap G \neq \emptyset \).

(2) There is a function \( r \in S^\omega \) such that \( r(i + 1) = M(r(i), v(i)) \), \( i < \omega \), and \( LN(r) \in F \).

To display this, suppose that (1) is fulfilled. From the construction of the function \( N \) it follows that \( r_s(i + 1) = M(r_s(i), v(i)) \), where \( r_s(i) = s \) if \( r(i) = s \) or \( r(i) = (a_0, \ldots, a_{n-1})s \). If \( (a_0, \ldots, a_{n-1}) \in \pi_1(LN(r) \cap G) \), with \( \pi_1 \) the 1st projection, then there is an index \( j \) such that \( a_j = A_j \in F \) and \( a_j \subseteq A_j \) or \( a_j = P \) for \( j' \neq j \). Suppose now that there is a second \( (b_0, \ldots, b_{n-1}) \in \pi_1(LN(r) \cap G) \) with \( a_j \neq b_j \) for some \( j \in n \). This implies immediately that \( (a_0, \ldots, a_{n-1}) \notin \pi_1(LN(r) \cap G) \), a contradiction. So it must be exactly \( LN(r) = A_j \). On the other hand, let (2) be satisfied with \( LN(r) = A_j \in F \). There is an integer \( k \) such that \( r(i) \in A_j \) for \( i \geq k \). Construct the function \( r' \in (S \cup T)^\omega \) by \( r'|k = r|k \),

\[
r'(k) = (\emptyset, \ldots, \emptyset)M(r(k - 1), v(k - 1))
\]

and

\[
r'(k + 1 + i) = H((\emptyset, \ldots, \emptyset)M(r(k - 1), v(k - 1), v(k + i))
\]

for \( v_k(i) : i + 1 \rightarrow \Sigma \) defined by \( v_k(i') = v(k + i') \) and \( H \) being the sequential extension of \( H \). For \( i \geq k \) we have \( r'(i) = (a_0, \ldots, a_{n-1})s \) provided \( a_j \subseteq A_j \). Here again by the second part of (2) we have \( (a_0, \ldots, a_{n-1}) \) with \( a_j = A_j \) belonging to \( \pi_1(LN(r')) \).

The above entails the identity \( E(\emptyset) = U(\emptyset) \), and thus, by Theorem 1, our thesis follows.

Now let \( \mathcal{N} = (S, M, S_0, F) \) be a l.d. automaton over \( \Sigma \). Define the automata \( \mathcal{N}_1(s) = (S, M, S_0, \{s\}, F) \), \( s \in F \). There exist d. automata \( \mathcal{N}_2(s) = (T, N, \{s\}, F) \), \( s \in F \), over \( \Sigma \) with \( N \subseteq M \). We have \( E(\mathcal{N}) = \bigcup_{s \in F} L(\mathcal{N}_1(s))E(\mathcal{N}_2(s)) \).

Since the regular sets concatenated with ED sets are again ED sets and ED sets are closed under the union, Theorem 2 will yield the following expansion result.

**Theorem 3.** Given a set \( A \subseteq \Sigma^\omega \). \( A \) is ED if and only if there are regular sets \( B_i \subseteq \Sigma^* \), and d. ED sets \( C_i \subseteq \Sigma^\omega \), \( i < n < \omega \), satisfying the identity \( A = \bigcup_{i < n} B_i \cup C_i \).

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**REFERENCES**


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