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ON THE STRUCTURE OF CERTAIN BOUNDED LINEAR OPERATORS[1]

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ABSTRACT. If every function f in the range of a bounded linear operator on L_p is equal to zero on a set of measure greater than a fixed number ϵ, it is shown that there is a common set of measure ϵ on which every function is zero. A decomposition theorem for such operators is proved.

1. **Introduction.** Consider the separable L_p space, $p \geq 1$, with finite measure denoted by $|\cdot|$. Throughout this paper all operators T are bounded linear operators mapping L_p into itself. Define for $f \in L_p$ the set $K(f) \equiv \{x|(Tf)(x) = 0\}$. We will establish that if $|K(f)| \geq \epsilon > 0$ for all $f \in L_p$, then there is a set K with $|K| \geq \epsilon$ for which $|K \cap K(f)| \geq \epsilon$ for all $f \in L_p$.

The problem of considering the sets $K(f)$ arises naturally when L_p is defined over a probability space (Ω, B, P). In this setting the sets $K(f)$ are events and the result establishes the existence of an event on which the range of T, $R(T)$, is zero almost everywhere.

2. **Main results and applications.** The principal result we wish to establish is the content of

Theorem 1. *If T is a bounded linear operator on L_p, and if $|K(f)| \geq \epsilon > 0$ for all $f \in L_p$, then there is a set K with $|K| \geq \epsilon$ such that $|K \cap K(f)| \geq \epsilon$ for all $f \in L_p$.*

Proof. Let f_1 and f_2 be in L_p. Define $f_\alpha = \alpha f_1 + f_2$. By hypothesis $|K(f_\alpha)| \geq \epsilon$. Each of the sets $K(f_\alpha)$ can be decomposed into two disjoint components, one consisting of $K(f_1) \cap K(f_2)$, and the other being the set $M(f_\alpha) \equiv \{x|\alpha(Tf_1)(x) = -(Tf_2)(x) \neq 0\}$, for $\alpha \neq 0$. The sets M_α are mutually exclusive and $\Sigma_{\alpha \in (-\infty,\infty)} |M_\alpha|) < \infty$. Thus, at most a countable number of the sets $M(f_\alpha)$ have positive measure. Let α^* be a number not in this countable set. Then $K(f_{\alpha^*}) = K(f_1) \cap K(f_2) \cup M(f_{\alpha^*})$. Since $|M(f_{\alpha^*})| = 0$, we must have $\epsilon \leq |K(f_{\alpha^*})| = |K(f_1) \cap K(f_2)|$. Thus, the result is established for the linear subspace of L_p spanned by f_1 and f_2.

Now define the function $f_{\alpha\beta} = \alpha f_1 + f_2 + \beta f_3$. As above, each $K(f_{\alpha\beta})$ can be decomposed into two disjoint components,

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$$M(f_{\alpha\beta}) = \{x \mid (\alpha(Tf_1) + (Tf_2))(x) = -\beta(Tf_3)(x) \neq 0\}$$

and $K(f_\alpha) \cap K(f_3)$. Thus there is a $\beta^* = \beta^*(\alpha)$ so that $|M(f_{\alpha\beta^*})| = 0$, and $\epsilon \leq |K(f_{\alpha\beta^*})| = |K(f_\alpha) \cap K(f_3)|$. Taking $\alpha = \alpha^*$ we have that $\epsilon \leq |K(f_{\alpha^*\beta^*})| = |K(f_1) \cap K(f_2) \cap K(f_3)|$. Proceeding by induction we conclude that $|\bigcap_{j=1}^n K(f_j)| \geq \epsilon$ for every set $\{f_1, \ldots, f_n\} \subseteq L_p$. Now let $f_1, f_2, \ldots$ be a basis for L_p. By the monotone convergence theorem, $|\bigcap_{j=1}^\infty K(f_j)| \geq \epsilon$. Define $K = \bigcap_{j=1}^\infty K(f_j)$. Any $f \in L_p$ has the representation $f = \Sigma a_i f_i$ and, since T is bounded, $Tf = \Sigma a_i T f_i$. So $K(f) \supseteq K$, and $|K \cap K(f)| \geq \epsilon > 0$, and the theorem is proved.

Corollary 1. *Let T satisfy the hypotheses of Theorem 1, with ϵ chosen as large as possible. Then $T = \chi_{K^c} T_1$, where T_1 is a bounded linear operator on L_p for which there exists a $f \in L_p$ such that $|\{x|(T_1 f)(x) = 0\}| = 0$. ($K^c$ is the complement of K.)*

Proof. Let K be the set constructed in Theorem 1. If we define $T_1 = \chi_K + T$ it is clear that for every $\delta > 0$ there is an $f \in L_p$ such that $|\{x|(T_1 f)(x) = 0\}| < \delta$. Suppose now that the following condition holds:

$$(*) \qquad \forall f \in L_p, \quad |K_1(f)| = |\{x \mid (T_1 f)(x) = 0\}| > 0.$$

Define S_n to be the subset of L_p consisting of f such that $|K_1(f)| \geq (1/n)$. A simple argument shows that S_n is closed. By (*), $\bigcup_n S_n = L_p$; therefore, some S_{n_0} contains an open ball with, say, center f_0. Because T is continuous, we can assert the existence of a smaller ball of radius δ' and center f_0 such that if $\|f - f_0\|_p < \delta'$, then $|K_1(f) \cap K_1(f_0)| \geq (1/3n_0) > 0$. For it is easy to see that

$$\|T\| \|f - f_0\|_p \geq \|Tf - Tf_0\|_p \geq \left(\int_M |(Tf_0)(x)|^p \, dx\right)^{1/p},$$

where $M = K_1(f) - (K_1(f) \cap K_1(f_0))$. Unless $|K_1(f) \cap K_1(f_0)|$ is large enough, the last integral above will exceed $\delta' \|T\|$. Thus, some S_{n_1} contains a ball about the origin and, hence, $S_{n_1} = L_p$.

From the proof of Theorem 1, we obtain

Corollary 2. *If T_1 is a bounded linear operator on L_p of a finite measure space, and if T_1 satisfies (*), then for every $f_1, f_2 \in L_p$ there is a constant α, as small as desired such that $|K_1(f_1 + \alpha f_2)| = |K(f_1) \cap K(f_2)|$.*

For an elementary application of Theorem 1, we remark that if two second order Gaussian stochastic processes $x_1(t)$ and $x_2(t)$ are related by a bounded linear operator T, that is, $x_1(t) = Tx_2(t)$ for each t, if T commutes with the resolution of identity induced by $x_2(t)$, and if T satisfies the conditions of Theorem 1, then $x_1(t)$ and $x_2(t)$ cannot have the same spectral type. (Cf. Hida [1] for the appropriate definitions of spectral type.)

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