PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 53, Number 2, December 1975

SOME FINITELY PRESENTED NON-3-MANIFOLD GROUPS

WOLFGANG H. HEIL

ABSTRACT. For every pair of integers (m, n), G. Baumslag and D. Solitar exhibited two-generator one-relator groups $G_{m,n}$ that are not residually finite or Hopfian. It is shown that these groups do not provide counterexamples to the conjecture that 3-manifold groups are residually finite (or at least Hopfian) by showing that they have subgroups which are not 3-manifold groups.

Searching for finitely presented non-Hopfian or nonresidually finite groups that are fundamental groups of 3-manifolds, W. Jaco [5, Question 13] asks if the two-generator one-relator groups $G_{m,n} = \{a, b: a^{-1}b^m a = b^n\}$ (m, n integers) are fundamental groups of 3-manifolds. These groups are Hopfian if and only if m and n have the same prime divisors or m or n divides the other [1]. Furthermore, if m or n divides the other, then $G_{m,n}$ is residually finite. In this note we answer the question for $G_{m,n}$ and some related groups.

Proposition 1. *Let m, n be integers with $|m| \neq |n|$, $|m| \neq 1$, $|n| \neq 1$. Then for $k \geq 3$ the group*

$$H_k = \{s_1, \ldots, s_k: s_j^m = s_{j+1}^n, \ j = 1, \ldots, k-1\}$$

is not the fundamental group of a 3-manifold.

Proof. (a) Suppose there exists a compact, orientable, irreducible and sufficiently large 3-manifold M such that $\pi_1(M) = H_k$. Let $d = \text{g.c.d.}(m, n)$, $m = dm'$, $n = dn'$; then $|m'| \neq |n'|$. Writing H_k as a free product of H_{k-1} and $\{s_k\}$ with amalgamation over $\{s_k^n\}$, it follows that s_1^α generates a unique maximal cyclic normal subgroup, where $\alpha = d(m')^{k-1}$. In fact, $\{s_1^\alpha\}$ is the center of H_k. By Waldhausen's theorem [7], M is a Seifert fiber space with a unique fiber which represents the element $h = s_1^\alpha \in \pi_1(M)$. Furthermore, since M is irreducible and the deficiency $\operatorname{def} \pi_1(M) = 1$, M has nonempty boundary by [2, Lemma 3.1]. Thus looking at the presentations (1.1) and (1.2) of [8], we find that $\pi_1(M)/\langle h\rangle$ is the free product of $r + 2p - 1$ (resp. $r + k - 1$) infinite cyclic and q finite cyclic groups (where r is the number of boundary components, p (resp. k) is the genus of the orbit surface, and q is the number of exceptional fibers). However $H_k/\langle s_1^\alpha\rangle$ is not of this

Received by the editors July 16, 1974 and, in revised form, October 18, 1974.
AMS (MOS) subject classifications (1970). Primary 57A10.

form, a contradiction. (This follows from the fact that a free product with amalgamation of indecomposable groups is indecomposable.)

(b) Suppose there exists an orientable 3-manifold M such that $\pi_1(M) = H_k$. Since H_k is finitely presented, there is a compact submanifold N of M such that $\pi_1(N) = H_k$ ([6]; also [4]). Thus N is orientable and, since H_k is not cyclic and not a free product, we can assume that N is irreducible (by capping off 2-spheres and replacing homotopy balls by balls). Since $H_1(N)$ is infinite, N is sufficiently large and we have case (a).

(c) Suppose there exists a nonorientable 3-manifold such that $\pi_1(M) = H_k$. As above, and by Scott's result, we can assume that M is compact and P^2-irreducible and that $k = 3$. Let K be the subgroup of H_3 generated by s_1^2, s_2^2, s_3^2. If m and n are both odd,

$$K = \{s_1^2, s_2^2, s_3^2: (s_1^2)^m = (s_2^2)^n, (s_2^2)^m = (s_3^2)^n\} \cong H_3,$$

K is the fundamental group of a compact submanifold N of the covering M' of M associated to K. Since M' covers the 2-fold orientable cover $\tilde{M}$ of M, it follows that N is orientable and we have a contradiction to case (a).

If m is even, $m = 2m'$, then

$$K = \{s_1^2, s_2, s_3: (s_1^2)^{m'} = s_2^n, s_2^m = s_3^n\} \quad \text{for } n \text{ odd},$$

and

$$K = \{s_1^2, s_2^2, s_3^2: (s_1^2)^{m'} = (s_2^2)^{n'}, (s_2^2)^{m'} = (s_3^2)^{n'}\} \quad \text{for } n = 2n'.$$

In either case, if $m \neq 2$, $n \neq 2$, the same argument as before applies.

Thus assume $m = 2$. If n is even, $n = 2l$, let ϕ: $H_3 \to Z_2$ be the homomorphism $\phi(s_i) = 1$, $i = 1, 2, 3$. By Reidemeister-Schreier,

$$\ker \phi = \{a, b, c: c^{-1}a^l c = a^l, b^{-1}a^{l^2}b = a^{l^2}\},$$

where $a = s_3^2$, $b = s_1 s_2$, $c = s_2^{-1}s_3$. The subgroup of $\ker\phi$ generated by a, b^2, c^2 is isomorphic to $\ker\phi$ and contained in $\pi_1(\tilde{M})$. It follows that there is a compact, orientable, irreducible 3-manifold N such that $\pi_1(N) = \ker\phi$. The center of $\pi_1(N)$ is generated by a^{l^2}. As in case (a), N would be a Seifert fiber space, but $\pi_1(N)/\langle a^{l^2}\rangle$ is not the right group.

If n is odd then $\langle s_1^2, s_2^2, s_3^2\rangle = \langle s_1^2, s_2, s_3\rangle$ has presentation $\{s_2, s_3, t_1, t_2: s_2^n = t_1^n, s_2^2 = s_3^n, t_1^2 = t_2^n\}$ where $t_1 = s_1 s_2 s_1^{-1}$, $t_2 = s_1 s_3 s_1^{-1}$. Again, this group would be the group of an orientable Seifert fiber space with boundary and with fiber generated by s_3^α ($\alpha = n^2$), but the quotient mod the center is not a free product of cyclic groups.

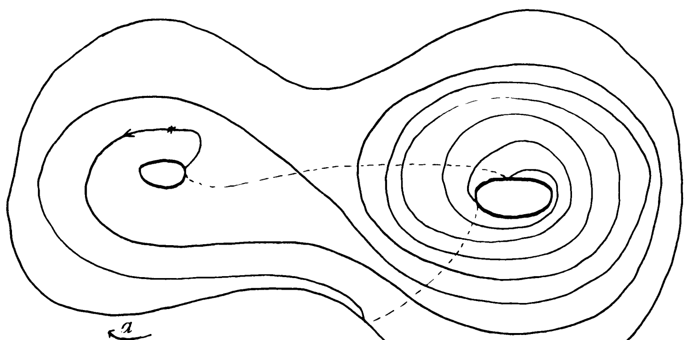
Remark. H_1 and H_2 are fundamental groups of 3-manifolds.

Proposition 2. *The group* $G_{m,n} = \{a, b: a^{-1}b^m a = b^n\}$ *is the fundamental group of a 3-manifold if and only if* $|m| = |n|$.

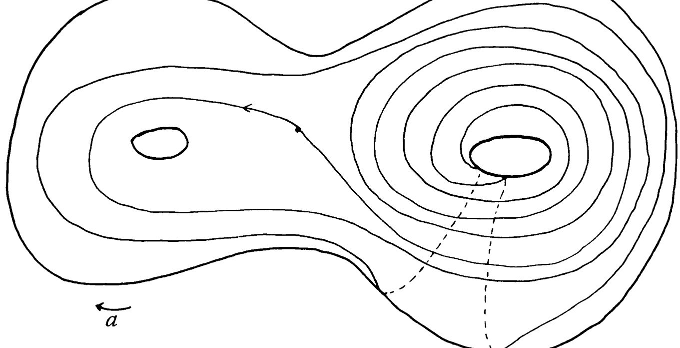
Proof. (a) The case $|m| = 1$, $|n| \neq 1$ is discussed in [5, p. 8.4].

Thus assume $|m| \neq |n|$ and $|m| \neq 1$, $|n| \neq 1$. Suppose there is a 3-manifold M such that $\pi_1(M) = G_{m,n}$. By Reidemeister-Schreier, the smallest normal subgroup H of $G_{m,n}$ generated by b has a presentation $H = \{s_j : s_j^m = s_{j+1}^n, -\infty < j < \infty\}$, where $s_j = a^j b a^{-j}$. Then the subgroup H_k of H generated by $s_1, \ldots, s_k$ is the fundamental group of a 3-manifold, which contradicts Proposition 1.

(b) For $|m| = |n|$ let V be a solid brezel of genus 2, let $\pi_1(V)$ be generated by simple closed curves a and b, and let c be a simple closed curve on ∂V which represents the element $a^{-1}b^m ab^{-m}$ (resp. $a^{-1}b^m ab^m$) $\in \pi_1(V)$ (the figure shows the case $m = 3$). Let M be the 3-manifold obtained from V by attaching a 2-handle along a regular neighborhood of c on ∂V. Then $\pi_1(M) = G_{m,n}$ ($m = n$, resp. $m = -n$).



$a^{-1}b^m ab$ b



$a^{-1}b^m ab^{-m}$ b

Another finitely presented non-Hopfian group was constructed by Higman [3]. It has generators a, b, c and relations $a^{-1}ca = b^{-1}cb = c^2$. It is a corollary to Proposition 2 that Higman's group is not the fundamental group of a 3-manifold. For, the subgroup generated by a and c is isomorphic to $G_{1,2}$.

It is interesting to note that for every integer m and n, the group $G_{m,n}$ acts freely on an irreducible 3-manifold. Let F be a torus with one boundary component, and a, b canonical generators for $\pi_1(F)$. In $F \times I$ identify regular neighborhoods of simple closed curves representing ab^n on $F_0 = F \times \{0\}$ and ab^m on $F_1 = F \times \{1\}$. The resulting 3-manifold M has group $\pi_1(M) = \{a, b, t: t^{-1}ab^m t = ab^n\}$. Then $G_{m,n}$ acts on the covering of M associated to $\langle a \rangle$ as group of covering translations.

REFERENCES

1. G. Baumslag and D. Solitar, *Some two-generator one-relator non-Hopfian groups*, Bull. Amer. Math. Soc. **68** (1962), 199–201. MR **26** #204.

2. D. B. A. Epstein, *Finite presentations of groups and 3-manifolds*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 205–212. MR **26** #1867.

3. G. Higman, *A finitely related group with an isomorphic proper factor group*, J. London Math. Soc. **26** (1951), 59–61. MR **12**, 390.

4. W. Jaco, *Finitely presented subgroups of three-manifold groups*, Invent. Math. **13** (1971), 335–346. MR **45** #9325.

5. ———, *The structure of 3-manifold groups* (preprint).

6. G. P. Scott, *Compact submanifolds of 3-manifolds*, J. London Math. Soc. 7 (1974), 246–250.

7. F. Waldhausen, *Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten*, Topology 6 (1967), 505–517. MR **38** #5223.

8. P. Orlik, E. Vogt and H. Zieschang, *Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten*, Topology 6 (1967), 49–64. MR 35 #3696.

DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FLORIDA 32306