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A NOTE ON CONTINUITY OF SEMIGROUPS OF MAPS

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ABSTRACT. An example is given of a separately continuous semigroup of transformations on Hilbert space which fails to be jointly continuous at $t = 0$.

1. Let X be a topological space and $\{F_t: t \geq 0\}$ a one-parameter semigroup of continuous maps of X into itself; that is, $F_{s+t} = F_s \circ F_t$ and $F_0 =$ identity. Suppose also that for each x in X the mapping $t \mapsto F_t(x)$ is continuous. Thus $(t, x) \mapsto F_t(x)$ is continuous in each variable separately. Under additional hypotheses we can conclude that this map is jointly continuous. Specifically, in [1] it is shown that if X is metrizable, then every point (t, x) with $t > 0$ is a point of joint continuity. Moreover, Dorroh [2] has shown that if X is locally compact and σ-compact then every point (t, x), including points with $t = 0$, is a point of joint continuity.

On the other hand, it was stated in [1] that there is an example for which joint continuity fails at $t = 0$, with X a certain subset of R^2. (There is a misprint in [1] on p. 1046; it is erroneously stated that such an example exists with $X = R^2$. This is, of course, ruled out by Dorroh's result.)

The aim of this note is to present an example illustrating failure of joint continuity at $t = 0$ with X a Hilbert space. Thus Dorroh's result does not generalize from finite-dimensional to infinite-dimensional manifolds.

2. Before giving the construction, it seems worthwhile to present a proof of joint continuity at $t = 0$ which is substantially more elementary than Dorroh's argument. We shall assume that the space X is locally compact and metrizable (rather than locally compact and σ-compact as in [2]).

Suppose that $\{F_t: t \geq 0\}$ is a separately continuous semigroup of maps on X. By [1] we have joint continuity for $t > 0$. We must establish joint continuity at $t = 0$. That is, given x in X and sequences $x_n \to x$, $t_n \to 0$, we have to show that $F_{t_n}(x_n) \to x$. If this is not the case, then there is a compact neighborhood K of x such that $F_{t_n}(x_n) \notin K$ for arbitrarily large n. Since $x_n \to x$, we may as well assume that $x_n \in K$, but $F_{t_n}(x_n) \notin K$, for all n. But then, because $F_t(x_n)$ is continuous in t, a connectedness argument

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shows that $F_{s_n}(x_n)$ lies on the boundary B of K for some s_n between 0 and t_n. Since B is compact, we may assume that $F_{s_n}(x_n)$ converges to a point y in B. Now obsetve that if $t > 0$,

$$F_t(y) = \lim F_t F_{s_n}(x_n) = \lim F_{t+s_n}(x_n) = F_t(x), \tag{1}$$

where joint continuity at (t, x) is used at the last step. If we finally let t converge to 0 in (1), we conclude that $y = x$. But this is a contradiction, since x is in the interior of K.

3. To prepare the ground for the construction of the Hilbert space example, we will exhibit the example on the subset of R^2 which we mentioned above.

We define the subset X as follows. Let p be the point $(1, 0)$. Introducing polar coordinates in the usual way, let $A = \{(r, \theta): r > 1, 0 < \theta < 2\pi\}$. Then let $X = A \cup \{p\}$. (Thus X is obtained from R^2 by deleting the positive x-axis and the closed unit disk, then restoring the point p. Note that X is locally compact except at p.)

Define $h(\theta) = \theta/(2\pi - \theta)$. The function h maps the interval $(0, 2\pi)$ homeomorphically onto $(0, \infty)$.

Now, for $t \geq 0$, define the map $F_t: X \to X$ in the following way. Set $F_t(p) = p$; and if $r > 1$, put $F_t(r, \theta) = (r, h^{-1}[h(\theta) + t/(r-1)])$. It is straightforward to check that $\{F_t: t \geq 0\}$ is a semigroup. (Roughly speaking, the flow F_t makes the circular arc with radius $r > 1$ collapse in a counterclockwise sense with increasing velocity as $r \to 1$.) To see that F_t is continuous for fixed $t > 0$, it is only necessary to worry about what happens at the point p. So let $x_n = (r_n, \theta_n)$ with $r_n \to 1$, $\theta_n \to 0$ or 2π. Then $h(\theta_n) + t/(r_n - 1) \to \infty$, and so $h^{-1}[h(\theta_n) + t/(r_n - 1)] \to 2\pi$. Hence, $F_t(r_n, \theta_n) \to (1, 2\pi) = p = F_t(p)$. If $t = 0$, $F_t =$ identity. Also, it is clear that $F_t(x)$ is continuous in t for fixed x in X.

Finally, we verify the failure of joint continuity at p with $t = 0$. Take $x_n = (r_n, \theta_n)$ with $r_n = 1 + 1/n$ and $\theta_n = 1/n$. Take $t_n = 1/n$. Then $x_n \to p$ and $t_n \to 0$, but

$$F_{t_n}(x_n) = (r_n, h^{-1}[h(1/n) + 1]) \to (1, h^{-1}(1)) = (1, \pi) \neq p. \tag{2}$$

We can now obtain an example on an open subset of Hilbert space quite cheaply (modulo infinite-dimensional topology!). Indeed, let H be a separable, infinite-dimensional Hilbert space. The space X considered above is easily seen to be a locally finite-dimensional simplicial complex; that is, the two-dimensional space X can be triangulated and homeomorphically embedded as a piecewise linear subset of H so that the vertices of the triangulation correspond to mutually orthogonal unit vectors. Hence, by [5, Theorem 3], the product space $X \times H$ is a manifold modelled on H. By the results in [3], $X \times H$ is homeomorphic to an open subset of H. We then simply take as our semigroup on $X \times H$ the maps $G_t = F_t \times I$.

The referee has observed that a simple modification of this construction yields a semigroup acting on the whole space H. Consider the metric cone C of X: if X is embedded as a piecewise linear subset of H, then C is the subset of $H \times \mathbf{R}$ consisting of the points $(\lambda x, 1-\lambda)$ with $x \in X$ and $0 \leq \lambda \leq 1$. The semigroup F_t extends in the obvious way to C: we define $F_t'(\lambda x, 1-\lambda) = (\lambda F_t(x), 1-\lambda)$. Then $F_t' \times I$ is the desired semigroup on $C \times H$. The point is that C is a *contractible* locally finite-dimensional simplicial complex, and so, by [3, Corollary 3], $C \times H$ is homeomorphic to H.

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