A NOTE ON CONTINUITY OF SEMIGROUPS OF MAPS

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ABSTRACT. An example is given of a separately continuous semi-
group of transformations on Hilbert space which fails to be jointly con-
tinuous at \( t = 0 \).

1. Let \( X \) be a topological space and \( \{ F_t : t \geq 0 \} \) a one-parameter semi-
group of continuous maps of \( X \) into itself; that is, \( F_{s+t} = F_s \circ F_t \) and
\( F_0 = \text{identity} \). Suppose also that for each \( x \) in \( X \) the mapping \( \langle t, x \rangle \mapsto F_t(x) \) is
continuous. Thus \( \langle t, x \rangle \mapsto F_t(x) \) is continuous in each variable separately.
Under additional hypotheses we can conclude that this map is jointly con-
tinuous. Specifically, in \([1]\) it is shown that if \( X \) is metrizable, then every
point \( (t, x) \) with \( t > 0 \) is a point of joint continuity. Moreover, Dorroh \([2]\)
has shown that if \( X \) is locally compact and \( \sigma \)-compact then every point
\( (t, x) \), including points with \( t = 0 \), is a point of joint continuity.

On the other hand, it was stated in \([1]\) that there is an example for
which joint continuity fails at \( t = 0 \), with \( X \) a certain subset of \( \mathbb{R}^2 \). (There
is a misprint in \([1]\) on p. 1046; it is erroneously stated that such an exam-
ple exists with \( X = \mathbb{R}^2 \). This is, of course, ruled out by Dorroh's result.)

The aim of this note is to present an example illustrating failure of
joint continuity at \( t = 0 \) with \( X \) a Hilbert space. Thus Dorroh's result does
not generalize from finite-dimensional to infinite-dimensional manifolds.

2. Before giving the construction, it seems worthwhile to present a
proof of joint continuity at \( t = 0 \) which is substantially more elementary than
Dorroh's argument. We shall assume that the space \( X \) is locally compact
and metrizable (rather than locally compact and \( \sigma \)-compact as in \([2]\)).

Suppose that \( \{ F_t : t \geq 0 \} \) is a separately continuous semigroup of maps
on \( X \). By \([1]\) we have joint continuity for \( t > 0 \). We must establish joint
continuity at \( t = 0 \). That is, given \( x \) in \( X \) and sequences \( x_n \rightarrow x \), \( t_n \rightarrow 0 \),
we have to show that \( F_{t_n}(x_n) \rightarrow x \). If this is not the case, then there is a
compact neighborhood \( K \) of \( x \) such that \( F_{t_n}(x_n) \not\in K \) for arbitrarily large \( n \).
Since \( x_n \rightarrow x \), we may as well assume that \( x_n \in K \), but \( F_{t_n}(x_n) \not\in K \), for all
\( n \). But then, because \( F_t(x_n) \) is continuous in \( t \), a connectedness argument

Received by the editors December 14, 1974.
AMS (MOS) subject classifications (1970). Primary 20M20, 54H15; Secondary
57A20.

1 Research partially supported by NSF grant GP-30798X.

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shows that \( F_{s_n}(x_n) \) lies on the boundary \( B \) of \( K \) for some \( s_n \) between 0 and \( t_n \). Since \( B \) is compact, we may assume that \( F_{s_n}(x_n) \) converges to a point \( y \) in \( B \). Now observe that if \( t > 0 \),

\[
F_t(y) = \lim_{n \to \infty} F_{t+s_n}(x_n) = \lim_{n \to \infty} F_{t+s_n}(x_n) = F_t(x),
\]

where joint continuity at \((t, x)\) is used at the last step. If we finally let \( t \) converge to 0 in (1), we conclude that \( y = x \). But this is a contradiction, since \( x \) is in the interior of \( K \).

3. To prepare the ground for the construction of the Hilbert space example, we will exhibit the example on the subset of \( R^2 \) which we mentioned above.

We define the subset \( X \) as follows. Let \( p \) be the point \((1, 0)\). Introducing polar coordinates in the usual way, let \( \Delta = \{(r, \theta): r > 1, 0 < \theta < 2\pi\} \). Then let \( X = \Delta \cup \{p\} \). (Thus \( X \) is obtained from \( K \) by deleting the positive x-axis and the closed unit disk, then restoring the point \( p \). Note that \( X \) is locally compact except at \( p \).)

Define \( h(\theta) = \theta/(2\pi - \theta) \). The function \( h \) maps the interval \((0, 2\pi)\) homeomorphically onto \((0, \infty)\).

Now, for \( t \geq 0 \), define the map \( F_t: X \to X \) in the following way. Set \( F_t(p) = p \); and if \( r > 1 \), put \( F_t(r, \theta) = (r, h^{-1}[h(\theta) + t/(r - 1)]) \). It is straightforward to check that \( \{F_t: t \geq 0\} \) is a semigroup. (Roughly speaking, the flow \( F_t \) makes the circular arc with radius \( r > 1 \) collapse in a counterclockwise sense with increasing velocity as \( r \to 1 \).) To see that \( F_t \) is continuous for fixed \( t > 0 \), it is only necessary to worry about what happens at the point \( p \). So let \( x_n = (r_n, \theta_n) \) with \( r_n \to 1 \), \( \theta_n \to 0 \) or \( 2\pi \). Then \( h(\theta_n) + t/(r_n - 1) \to \infty \), and so \( h^{-1}[h(\theta_n) + t/(r_n - 1)] \to 2\pi \). Hence, \( F_t(r_n, \theta_n) \to (1, 2\pi) = p = F_t(p) \). If \( t = 0 \), \( F_t = \) identity. Also, it is clear that \( F_t(x) \) is continuous in \( t \) for fixed \( x \) in \( X \).

Finally, we verify the failure of joint continuity at \( p \) with \( t = 0 \). Take \( x_n = (r_n, \theta_n) \) with \( r_n = 1 + 1/n \) and \( \theta_n = 1/n \). Take \( t_n = 1/n \). Then \( x_n \to p \) and \( t_n \to 0 \), but

\[
F_{t_n}(x_n) = (r_n, h^{-1}[h(1/n) + 1]) \to (1, h^{-1}(1)) = (1, n) \neq p.
\]

We can now obtain an example on an open subset of Hilbert space quite cheaply (modulo infinite-dimensional topology!). Indeed, let \( H \) be a separable, infinite-dimensional Hilbert space. The space \( X \) considered above is easily seen to be a locally finite-dimensional simplicial complex; that is, the two-dimensional space \( X \) can be triangulated and homeomorphically embedded as a piecewise linear subset of \( H \) so that the vertices of the triangulation
correspond to mutually orthogonal unit vectors. Hence, by [5, Theorem 3], the product space $X \times H$ is a manifold modelled on $H$. By the results in [3], $X \times H$ is homeomorphic to an open subset of $H$. We then simply take as our semigroup on $X \times H$ the maps $G_t = F_t \times I$.

The referee has observed that a simple modification of this construction yields a semigroup acting on the whole space $H$. Consider the metric cone $C$ of $X$: if $X$ is embedded as a piecewise linear subset of $H$, then $C$ is the subset of $H \times \mathbb{R}$ consisting of the points $(\lambda x, 1 - \lambda)$ with $x \in X$ and $0 \leq \lambda \leq 1$. The semigroup $F_t$ extends in the obvious way to $C$: we define $F'_t(\lambda x, 1 - \lambda) = (\lambda F_t(x), 1 - \lambda)$. Then $F'_t \times I$ is the desired semigroup on $C \times H$. The point is that $C$ is a contractible locally finite-dimensional simplicial complex, and so, by [3, Corollary 3], $C \times H$ is homeomorphic to $H$.

Acknowledgement. I wish to thank Professor Victor Klee for some interesting correspondence concerning the proof that $X \times H$ is homeomorphic to an open subset of $H$. In particular, Klee has shown that this can be obtained via the machinery developed in [4].

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