CERTAIN MULTIPLE VALUED HARMONIC FUNCTION

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Abstract. The difference equation \( v(x^+) - v(x) = u(x) \) is solved for any harmonic \( u \) in the covering space of an unknotted curve.

The purpose of this note is to answer a problem posed by H. Lewy [2]. The situation is as follows: Given is a curve \( \Gamma \) diffeomorphic, in \( \mathbb{R}^3 \), to a circle. That is, we have a diffeomorphism \( \varphi: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) that maps \( \Gamma \) onto a circle \( C \).

If we now consider the universal covering space \( S \), of \( \mathbb{R}^3 \sim \Gamma \), it is asked if given a function \( u \), harmonic on \( S \), we can find a harmonic function \( \nu \), such that \( \nu(x^+) - \nu(x) = u(x) \) (where \( x^+ \) denotes the point obtained from \( x \) in \( S \) following a path \( \delta \) whose projection \( \pi(\delta) \) in \( \mathbb{R}^3 \sim \Gamma \) is a closed curve that loops \( \Gamma \) once). We are going to prove that such a function \( \nu \) exists.

Proof. The proof is a modification of the proof in [2], to make an integral convergent.

More precisely, let us consider, as in [2], the disk \( D \) in \( \mathbb{R}^3 \) that has \( C \) as boundary and let \( \sigma = \varphi^{-1}(D) \). We will consider neighborhoods \( U_n \) of \( \Gamma \),

\[ (1) \quad U_n = \{ x : d(x, \Gamma) \leq 2^{-n} \}. \]

By means of the diffeomorphism, for \( n_0 \) large enough we can construct a continuous mapping \( \tilde{x}(x): U_{n_0} \rightarrow \Gamma \) such that

\[ (2) \quad |\tilde{x}(x) - x| \leq Kd(x, \Gamma) \quad (K \text{ a constant}). \]

\( B_1 \) will denote a ball verifying \( (U_0 \cup \sigma) \subset B_1 \).

Let us consider now the two consecutive leaves \( S_1, S_2 \) of \( S \) that are between the copies \( \sigma_0 \) and \( \sigma_2 \) of \( \sigma \) on \( S \) and have the copy \( \sigma_1 \) as common boundary.

Then, in the compact set

\[ W_n = \left[ (S_1 \cup S_2) \cap \pi^{-1}(B_1) \right] \sim \pi^{-1} U_n \]

there exists a \( \lambda_n \) such that

\[ (3) \quad |u| \leq \lambda_n. \]

(As before \( \pi \) denotes the canonical projection \( \pi: S \rightarrow \mathbb{R}^3 \sim \Gamma \).) Any first derivative of \( u \) is therefore bounded in \( W_n \) by some \( \lambda_n \'). Let

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\(\bar{\lambda}_n = \max(\lambda_n, \lambda'_n).\)

We are going to consider the function

\[
\nu(x) = \frac{1}{4\pi} \int_{y \in \mathcal{U}_n} \left( u(y) \frac{\partial}{\partial y} V - V \frac{\partial}{\partial y} u(y) \right) \, d\sigma(y)
\]

\[+ \sum_{k=n_0}^{\infty} \int_{y \in \mathcal{U}_k} \left( u(y) \frac{\partial}{\partial y} (V - V_k)(x, y) - (V - V_k)(x, y) \frac{\partial}{\partial y} u(y) \right) \, d\sigma(y)
\]

where \(V(x, y) = 1/|x - y|\) and \(V_k\) is a suitable Taylor’s polynomial of \(V\) around the point \(\bar{y}\) (as defined in (2)). The values of \(u\) are taken over the copy \(\sigma_1\) of \(\sigma\), bounding \(S_1\) from \(S_2\). Let us then recall the following developments:

\[
\frac{1}{|x - y|} = \frac{1}{|x - y_0|} \left( \sum_{n=0}^{\infty} P_n(\cos \theta) \left( \frac{|y - y_0|}{|x - y_0|} \right)^n \right)
\]

convergent for \(|y - y_0| < |x - y_0|\), \(\sup_{|u| \leq 1} |P_n(u)| = 1\) with \(\theta\) the angle between \(x - y_0\) and \(y - y_0\).

\[
\nabla_y \frac{1}{|x - y|} = \frac{1}{|x - y_0|^3} \sum_{n=1}^{\infty} \left( P_{n-1}(u)(y - y_0) - P_n'(u) \frac{|y - y_0|}{|x - y_0|}(x - y_0) \right) \left( \frac{|y - y_0|}{|x - y_0|} \right)^{n-2}
\]

with \(u = \cos \theta\) again convergent for \(|y - y_0| < |x - y_0|\), \(\sup_{|u| \leq 1} |P_n'(u)| \leq n(n + 1)/2\).

Each term of (6) and (7) is a harmonic polynomial (see [1, pp. 124 and 142]).

Turning back to (5) we choose \(V_k\) to be the development (6) with \(y_0 = \bar{y}(y)\) up to an order \(l(k)\) verifying

\[
\lambda_k 2^{-l(k)/2} < 2^{-k}.
\]

Then if \(x \not\in \mathcal{U}_m \cup \sigma_1\) and \(B(x)\) is a ball centered at \(x\) with \(\overline{B(x)} \cap (\mathcal{U}_m \cup \sigma_1) = \emptyset\), for \(y \in U_k\), with \(k \geq \max[K(m + 1), n_0]\), we have, for any \(x' \in B(x)\),

\[
(V - V_k)(x', y) \leq \frac{C_2^{-l(k)/2}}{|x' - \bar{y}|} \leq C_2^m 2^{-l(k)}
\]

and

\[
|\nabla(V - V_k)(x', y)| \leq 2^{3m} C_l(2)_k^{-l(k)} \leq C' 2^{-l(k)/2}
\]

where \(C\) and \(C'\) denote constants. Hence, the first \(\max[K(m + 1), n_0]\) terms of (5) are bounded harmonic functions on \(B\), and the remaining terms give us, on \(B\), an absolutely and uniformly convergent series of harmonic functions. Now, if \(B\) is a small ball intersecting \(\sigma\) (with \(\overline{B} \cap \Gamma = \emptyset\)), and we remove from (5) the integral.

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(11) \[
\frac{1}{4\pi} \int_{B \cap \sigma} \left( u(y) \frac{\partial}{\partial y_j} V(x,y) - V(x,y) \frac{\partial}{\partial y_j} u(y) \right) d\sigma
\]

(which involves only a finite number of terms of (5)), the remaining terms give us again an absolutely and uniformly convergent series of harmonic functions in any closed subball of \( B \). But as in [2], (11) gives us the desired jump in \( u(x) \) of the function \( v \) across \( \sigma \), and that completes the proof.

REFERENCES


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