SIMULTANEOUS SPLINE APPROXIMATION AND INTERPOLATION PRESERVING NORMS


Abstract. In this paper, it is proved that splines of order $k$ ($k > 2$) have property SAIN. The proof of this result is based on the important properties of B-splines.

1. Introduction. In a recent manuscript [5], Lambert proved that the twice continuously differentiable cubic splines possess property SAIN (simultaneous approximation and interpolation which is norm preserving) on $C[a, b]$ where the interpolatory constraints are point evaluations. In this paper we establish the more general result for splines of any order greater than 1 while at the same time supplying a simple proof. More precisely, we will show

**Theorem 1.** Splines of order $k$ (degree $k - 1$) and continuity class $C^{k-2}[a, b]$ possess property SAIN on $C[a, b]$ with respect to a finite number of point evaluations.

Deutsch and Morris [2] introduced the property SAIN:

**Definition.** Let $X$ be a normed linear space, $M$ a dense subset of $X$, and $V$ a finite dimensional subspace of $X^*$. The triple $(X, M, V)$ has property SAIN if, for every $x \in X$ and $\varepsilon > 0$, there exists $y \in M$ such that

$$
\|x - y\| < \varepsilon, \quad \|x\| = \|y\|, \quad \text{and} \quad \gamma(x) = \gamma(y)
$$

for every $y \in \Gamma$.

This definition was motivated by the work of Wolibner [6] and Yamabe. (See [2] for references.) Also Lambert [4] has studied property SAIN for $L_1$ and $C(T)$, and Holmes and Lambert [3] studied the property SAIN from a geometrical point of view.

2. Proof of Theorem 1. Let $S^k$ denote the set of splines of order $k$ and continuity class $C^{k-2}$ with a finite number of knots in $[a, b]$. Further set $\Gamma = \text{span} \{\delta_{a}, \ldots, \delta_{b}\}$ where $\delta_x$ represents the usual point evaluation functional at $x$ which is an element of $C[a, b]^*$. We now show that $(C[a, b], S^k, \Gamma)$ has property SAIN.

Our proof relies heavily on the fundamental properties of B-splines. Following de Boor [1], we denote by $N_{i,k}$ the normalized B-spline of order $k$ supported on $[t_i, t_{i+k}]$ where $\{t_i\}_{i=0}^{N}$ is a partition of $[a, b]$, and where $N_{i,k}(t) = (t_{i+k} - t_i)[t_i, \ldots, t_{i+k}]_S(S - t)^{k-1}_+$ with $[t_i, \ldots, t_{i+k}]_S$ denoting the...
kth divided difference operator in the variable $S$. Recall the following important properties of the $N_{i,k}$:

(i) $N_{i,k}(t) \geq 0$ for all $t$,
(ii) $\text{supp } N_{i,k} = [t_i, t_{i+k}]$, and
(iii) $\sum_{j=i-j+1-k} N_{i,k}(t) = 1$, $t \in [t_j, t_{j+1}]$.

(For a detailed account of B-splines, see [1].) Let $\pi$ denote a partition of $[a, b]$, i.e., $\pi = \{t_i\}_{i=0}^N$ with $a = t_0 < \cdots < t_N = b$. The mesh size of $\pi$ will be denoted by $|\pi| = \max_{0 \leq i \leq N-1} |t_{i+1} - t_i|$. In order for the $N_{i,k}$'s to be a basis of the $C^{k-2}$ (order $k$) splines, with knots only in $\pi$, it is necessary to create $k - 1$ knots on the left of $t_0$ and $k - 1$ knots on the right of $t_N$. Given a partition $\pi$ of $[a, b]$ we define $\tilde{\pi}$ as the partition $t_{-k+1} < \cdots < t_{N+k-1}$ where the extra points are chosen so that $|\tilde{\pi}| = |\pi|$. Proceeding with the proof, let $f \in C[a, b]$ be given along with interpolation points $\{t_i\}_{i=1}^N$ and $\epsilon > 0$. Without loss of generality assume that $f$ attains its norm at some $\tau_i$. It remains to display a spline of the correct type which interpolates $f$ at the $t_i$ and

$$\|f - g\| = \max_{-k+1 \leq i \leq N-1} \max_{t \in B_i} |\alpha_i - f(t)| N_{i,k}(t) = \max_{-k+1 \leq i \leq N-1} \max_{t \in B_i} |\alpha_i - f(t)| N_{i,k}(t) = \max_{-k+1 \leq i \leq N-1} \max_{t \in B_i} |\alpha_i - f(t)| N_{i,k}(t) \leq \omega(f, 2k \epsilon) \leq 2k \omega(f, \epsilon) \leq \epsilon,$$

where $B_i = [t_i, t_{i+k}] \cap [a, b]$. This completes the proof of the theorem.

Remark. In the course of the proof of Theorem 1, we add, if necessary, an additional interpolation constraint at a point where $f$ attains its norm. This is to insure that $g$ satisfies the norm preservation property. However, even without adding this additional interpolation constraint, we still have $\|g\| \leq \|f\|$, and the theorem then follows by applying Lemmas 2.1 and 2.2 of [2].

There are many ways to extend Theorem 1. For instance, one can consider the natural splines

$$S_{0}^{2k} = \{s \in S^{2k} : s^{(j)}(a) = s^{(j)}(b) = 0, k \leq j \leq 2k - 2\}$$

and obtain the following:

**Corollary.** The natural splines $S_{0}^{2k}$ have the property $SA1N$.

To prove this result, we just alter the construction above to insure that $\alpha_{-k+1} = \cdots = \alpha_0$ and $\alpha_{N-1} = \cdots = \alpha_{N-k}$, so that $g^{(j)}(a) = g^{(j)}(b) = 0$ for $j = k, \ldots, 2k - 2$. 


References


5. ———, *Simultaneous approximation and interpolation which preserves the norm by cubic splines in $C[a,b]$* (submitted).


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