TIETZE-TYPE THEOREMS ON MONOTONE INCREASING SETS

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ABSTRACT. The Tietze theorem on convex sets is generalized to monotone increasing sets and strictly monotone increasing sets, which include convex sets as a special case. The main theorem is that a closed connected set in $E_2$ is monotone increasing if and only if it is locally monotone increasing. A similar result is proved for strictly monotone increasing sets.

1. Introduction and preliminaries. In 1928 Tietze [3] proved that a closed connected set in $E_n$ is convex if and only if it is locally convex. Klee [1] generalized this theorem to a topological linear space. We will prove that a closed connected set in $E_2$ is monotone increasing if and only if it is locally monotone increasing. A similar result is proved for strictly monotone increasing sets.

Let $E_2$ be a two-dimensional Euclidean space with the rectangular coordinate axes. The closure, interior, boundary, and convex hull of a set $S$ in $E_2$ are denoted by $\text{cl} S$, $\text{int} S$, $\text{bd} S$, $\text{conv} S$, respectively. If $x$ and $y$ are distinct points, then $xy$ denotes the closed line segment joining $x$ and $y$ and $\text{int} xy$ denotes the relative interior of $xy$. For two distinct points $x$ and $y$ in $E_2$ not lying on a vertical line, let $m(x,y) = m(L(x,y))$ denote the slope of the line $L(x,y)$ through $x$ and $y$. If $x$ and $y$ are two points in $E_2$ not lying on a vertical or horizontal line, the line segment $xy$ determines two closed convex triangles having $xy$ as the hypotenuse and each of the remaining sides parallel to one of the axes. The triangle lying below $xy$ is denoted by $T(x,y)$, called the lower triangle determined by $x$ and $y$. A convex arc $C(x,y)$ joining $x$ and $y$ is called a monotone increasing arc if $m(x,y) > 0$ and $C(x,y) \subset T(x,y)$.

DEFINITION 1. A set $S$ in $E_2$ is monotone increasing if for each pair of distinct points $x \in S$, $y \in S$, it is true that

1. there exists a monotone increasing arc $C(x,y)$ in $S$ joining $x$ and $y$ if $m(x,y) > 0$; and

2. $xy \subset S$ if $x$ and $y$ are on a vertical line or if $m(x,y) \leq 0$.

We note that the arc $C(x,y)$ in (1) of Definition 1 above can contain at most one horizontal line segment since $C(x,y)$ is a convex arc and $C(x,y) \subset T(x,y)$.

DEFINITION 2. A set $S$ in $E_2$ is strictly monotone increasing if for each pair of distinct points $x \in S$, $y \in S$, it is true that

1. if $m(x,y) > 0$, there exists a monotone increasing arc $C(x,y)$ joining $x$
and $y$ in $S$ where $C(x,y)$ contains no horizontal line segment; and

(2) $xy \subset S$ if $x$ and $y$ are on a vertical line or if $m(x,y) \leq 0$.

**Definition 3.** A set $S$ in $E_2$ is locally (strictly) monotone increasing at a point $z \in S$ if there exists a neighborhood $N$ of $z$ such that $N \cap S$ is (strictly) monotone increasing. A set $S$ is locally (strictly) monotone increasing if it is locally (strictly) monotone increasing at each of its points.

We observe that if $S \subset E_2$ is an open monotone increasing set, then $S$ is strictly monotone increasing. To see this, let $x$ and $y$ be arbitrary distinct points of $S$. If $x$ and $y$ lie on a vertical line or if $m(x,y) \leq 0$, then $xy \subset S$. Suppose $m(x,y) > 0$, and suppose that $C(x,y)$ is a monotone increasing arc joining $x$ and $y$ in $S$ with $y$ lying in the upper half-plane bounded by the horizontal line through $x$. Let $xu$ be the longest horizontal line segment contained in $C(x,y)$. Since $S$ is open, for each $z \in xu$, there exists an open set $N_z$ of $z$ such that $N_z \subset S$. The compactness of the line segment $xu$ implies the existence of a finite collection of these open sets, say $N_{z_1}, \ldots, N_{z_n}$, with $xu$ contained in $N_{z_1} \cup \cdots \cup N_{z_n} \subset S$. Without loss of generality, we assume that $u \in N_{z_n}$. Now we choose a point $w$ in $N_{z_n} \cap C(u,y)$, where $C(u,y)$ is the subarc of $C(x,y)$ joining $u$ and $y$, such that $w \neq u, w \neq y$ and $xw \subset N_{z_1} \cup \cdots \cup N_{z_n}$. The set $xw \cup C(w,y)$, where $C(w,y)$ is the subarc of $C(x,y)$ joining $w$ and $y$, is a monotone increasing arc containing no horizontal line segment and joining $x$ and $y$ in $S$. This proves that $S$ is strictly monotone increasing.

To prove our characterization theorems, we apply a result on stripwise subconvex sets [2]. The relevant definitions and theorem are stated below.

**Definition 4.** A set $S$ in $E_2$ is stripwise subconvex if for each pair of distinct points $x$ and $y$ in $S$ there exists a convex arc $A(x,y)$ in $S$ joining $x$ and $y$ such that

(1) $A(x,y) = xy$ if $x$ and $y$ are on a vertical line; and

(2) if $x$ and $y$ are not on a vertical line, then $A(x,y)$ lies in the closed lower half-plane bounded by the line $L(x,y)$ and $A(x,y)$ lies in the convex hull of the vertical lines through $x$ and $y$ respectively. Such an arc $A(x,y)$ is called a stripwise subconvex arc.

**Definition 5.** A set $S$ in $E_2$ is locally stripwise subconvex if for each $z \in S$ there exists a neighborhood $N$ of $z$ such that $N \cap S$ is stripwise subconvex.

**Theorem 1.** A closed connected set $S$ in $E_2$ is stripwise subconvex if and only if it is locally stripwise subconvex [2].

We note that a monotone increasing set is stripwise subconvex but the converse is not true in general.

If $S \subset E_2$ is a closed set and two points $x$ and $y$ in $S$ can be joined by a stripwise subconvex arc in $S$, then there exists a minimal stripwise subconvex arc $C_0(x,y)$ joining $x$ and $y$ in $S$ in the sense that if $C(x,y)$ is any other stripwise subconvex arc joining $x$ and $y$ in $S$, it is true that $\text{conv } C_0(x,y) \subset \text{conv } C(x,y)$. The same assertion holds for monotone increasing arcs. To see this, first consider the case when $xy \subset S$. Let $(C(x,y))$ be the collection of all stripwise subconvex (monotone increasing) arcs joining $x$ and $y$ in $S$, this collection is nonempty by hypothesis. Now set

$$C_0(x,y) = \text{bd } \left\{ \cap \text{conv } C(x,y) \right\} - \text{intv } xy$$

where the intersection is taken over all members of the collection. Since the
set $S$ is closed, we have $C_0(x,y) \subseteq S$. Clearly $C_0(x,y)$ is a minimal stripwise subconvex (monotone increasing) arc joining $x$ and $y$ in $S$. In case $xy \subseteq S$, then $xy$ is the minimal arc required.

The theorem stated below will be useful in the following discussion. A more general form of this theorem can be found in Valentine [4].

**Theorem 2.** Let $S$ be a closed convex set in $E_2$. Each compact connected portion of the boundary of $S$ which is not contained in a line segment contains an exposed point of $S$. If $S$ is a line segment, it has two exposed points.

2. The results.

**Theorem 3.** Let $S$ be a closed connected set in $E_2$. The set $S$ is monotone increasing if and only if it is locally monotone increasing.

**Proof.** The necessity is obvious. To prove the sufficiency, let $x$ and $y$ be arbitrary distinct points in $S$. Since $S$ is locally monotone increasing, it is locally stripwise subconvex. By Theorem 1, $S$ is stripwise subconvex. If $x$ and $y$ lie on a vertical line, then $xy \subseteq S$. Otherwise, let $A(x,y)$ be a stripwise subconvex arc joining $x$ and $y$ in $S$. We may assume that $A(x,y)$ is minimal. Now we consider the following cases according to the slope $m(x,y)$.

(a) $m(x,y) > 0$. If $A(x,y) = xy$, there is nothing to prove. Suppose that $A(x,y) \neq xy$, and without loss of generality we may assume that $y$ lies in the upper half-plane bounded by the horizontal line through $x$. We will show that $A(x,y)$ is a monotone increasing arc joining $x$ and $y$ in $S$. Let $z$ be an exposed point of $\text{conv} A(x,y)$ on $A(x,y) \setminus \{x,y\}$ and let $L$ be a corresponding line of support to $\text{conv} A(x,y)$ through $z$ for which $L \cap \text{conv} A(x,y) = \{z\}$. We wish to show that $L$ has positive slope. Suppose on the contrary that $m(L) \leq 0$. Since $S$ is locally monotone increasing, a neighborhood $N$ of $z$ exists such that $N \cap S$ is monotone increasing. We may choose $N$ to be a convex neighborhood such that $x \notin N$ and $y \notin N$. See Figure.
Since $A(x,y)$ is compact, $\text{conv} A(x,y)$ is compact. We can choose points $u$ and $v$ on $L$ with $z \in \text{int} u v$, and neighborhoods $U$ of $u$ and $V$ of $v$ such that $U \subset N, \; V \subset N, \; U \cap \text{conv} A(x,y) = \emptyset, \; \text{and} \; V \cap \text{conv} A(x,y) = \emptyset$. The fact $A(x,y) \neq xy$ implies that $\text{int} \text{conv} A(x,y) \neq \emptyset$, thus

$$\text{cl} (\text{conv} A(x,y)) = \text{cl} (\text{int} \text{conv} A(x,y)).$$

Hence any neighborhood of $z$ contains an interior point of $\text{conv} A(x,y)$. Choose a point $t \in \text{int} \text{conv} A(x,y) \cap N$ sufficiently close to $z$ so that there exists a line $L_1$ through $t$ parallel to $L$ with $L_1 \neq L$ such that $L_1 \cap U \neq \emptyset$ and $L_1 \cap V \neq \emptyset$. Therefore $L_1 \cap \text{conv} A(x,y)$ contains no points of $U$ and $V$ and $L_1 \cap \text{conv} A(x,y) \subset N$. Let $r$ and $s$ denote $L_1 \cap \text{conv} A(x,y)$ where $r$ and $s$ are boundary points of $\text{conv} A(x,y)$ and, therefore, points of $A(x,y)$. The line segment $rs$ is nondegenerate since $t \in \text{int} rs$. We have $m(r,s) = m(L_1) = m(L) \leq 0$ since $L_1$ is parallel to $L$. Since $N \cap S$ is monotone increasing and $r$ and $s$ lie in $N \cap S$ with $m(r,s) \leq 0$, it follows that $rs \subset S$. Now set $B(x,y) = A(x,r) \cup rs \cup A(s,y)$ where $A(x,r)$ and $A(s,y)$ are the subarcs of $A(x,y)$ joining $x$ to $r$ and $s$ to $y$ respectively. Since $z$ is an exposed point of $\text{conv} A(x,y)$ and $z \in A(r,s)$ where $A(r,s)$ is the subarc of $A(x,y)$ joining $r$ and $s$, we have $A(r,s) \neq rs$. The arc $B(x,y)$ is a stripwise subconvex arc joining $x$ and $y$ in $S$. The existence of $B(x,y)$ contradicts the minimality of $A(x,y)$ since $\text{conv} A(x,y)$ is not contained in $\text{conv} B(x,y)$. Hence $L$ must have positive slope.

If $A(x,y)$ is not contained in the closed upper half-plane bounded by the horizontal line through $x$, let $w$ be the point of intersection of $A(x,y)$ and the horizontal line through $x$. Thus $A(x,w) - \{x,w\}$ lies in the open lower half-plane bounded by the horizontal line through $x$ where $A(x,w)$ is the subarc of $A(x,y)$ joining $x$ to $w$. Theorem 2 and this fact imply that there exists an exposed point $b$ of $\text{conv} A(x,w)$ on $A(x,w) - \{x,w\}$ with a line of support $K$ to $\text{conv} A(x,w)$ through $b$ such that $K \cap \text{conv} A(x,w) = \{b\}$ and $m(K) \leq 0$. Clearly the point $b$ is an exposed point of $\text{conv} A(x,y)$ and the line $K$ is a line of support to $\text{conv} A(x,y)$ through $b$ such that $K \cap \text{conv} A(x,y) = \{b\}$. The fact $m(K) \leq 0$ contradicts what we proved above. Hence $A(x,y)$ lies in the closed upper half-plane bounded by the horizontal line through $x$. But $A(x,y)$ is stripwise subconvex, thus $A(x,y)$ must lie in the lower triangle $T(x,y)$ and, hence, $A(x,y)$ is a monotone increasing arc joining $x$ and $y$ in $S$.

(b) $m(x,y) < 0$. For $z$ and $L$ described in (a), $L$ has positive slope. From Theorem 2 and the fact $m(x,y) < 0$, it follows that $A(x,y) = xy$.

(c) $m(x,y) = 0$. As in (a) we conclude that $A(x,y)$ must lie in the closed upper half-plane bounded by the horizontal line through $x$. Consequently $A(x,y) = xy$.

We conclude that $S$ is a monotone increasing set.

**Theorem 4.** Let $S$ in $E_2$ be a closed connected set. The set $S$ is strictly monotone increasing if and only if it is locally strictly monotone increasing.

**Proof.** The necessity is obvious. To prove the sufficiency, let $x$ and $y$ be arbitrary distinct points in $S$. By Theorem 3, $S$ is monotone increasing. Hence $xy \subset S$ if $x$ and $y$ are on a vertical line or if $m(x,y) \leq 0$. If $m(x,y) > 0$, then there exists a monotone increasing arc $C(x,y)$ joining $x$ and $y$ in $S$. Since $S$ is
closed, we may take $C(x,y)$ to be minimal. We only need to show that $C(x,y)$
contains no horizontal line segment. Assume that $y$ lies in the upper half-plane
bounded by the horizontal line through $x$. Suppose $xz$ is the longest horizontal
line segment of $C(x,y)$. Since $S$ is locally strictly monotone increasing at the
point $z$, there is a neighborhood $N$ of $z$ such that $N \cap S$ is strictly monotone
increasing. If $x \neq z$, there exists a point $s \in N \cap \text{int} xz$, and a point$t \in N \cap C(z,y)$ with $t \neq z$ and $t \neq y$ where $C(z,y)$ is the subarc of $C(x,y)$
joining $z$ and $y$. The points $s$ and $t$ lie in $N \cap S$, and $m(s,t) > 0$, therefore
there exists a monotone increasing arc $D(s,t)$ in $N \cap S$ such that $D(s,t)$
contains no horizontal line segment. Let
\[
D_0(s,t) = \text{bd} \left( \text{conv} \ D(s,t) \cap \text{conv} \ C(s,t) \right) - \text{int} \ s t
\]
where $C(s,t)$ is the subarc of $C(x,y)$ joining $s$ and $t$. It is obvious that
$D_0(s,t) \subset S$. Set $C_0(x,y) = xs \cup D_0(s,t) \cup C(t,y)$, where $C(t,y)$ is the sub-
arc of $C(x,y)$ joining $t$ and $y$. The arc $C_0(x,y)$ is clearly convex and it is a
monotone increasing arc joining $x$ and $y$ in $S$. But $\text{conv} \ C(x,y)$ is not
contained in $\text{conv} \ C_0(x,y)$ since $D(s,t)$ and $D_0(s,t)$ contain no horizontal line
segment. The existence of $C_0(x,y)$ contradicts the minimality of $C(x,y)$. Thus
$x = z$ and $C(x,y)$ contains no horizontal line segment. Hence $S$ is a strictly
monotone increasing set.

Bibliography

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