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THE AUTOMORPHISMS OF $O_4^+(V)$ IN THE ANISOTROPIC CASE

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ABSTRACT. The automorphisms of the rotation group of a quaternary anisotropic quadratic space defined over a field of characteristic not two are determined.

In this paper we determine the automorphisms of the rotation group $O_4^+(V)$ of a quaternary anisotropic quadratic space V defined over a field F of characteristic not two and thus resolve an open question mentioned by Dieudonné in his book *La géométrie des groupes classiques* (see [1, p. 101]). The automorphisms of $O_4^+(V)$ have been determined by Dieudonné in the case that V has positive Witt index and by Wonenburger in the case that the discriminant of V is a square. (See [1, pp. 101-103] and [3, p. 195] respectively.) Our approach treats the square discriminant and nonsquare discriminant cases simultaneously, and it is free of any reference to the Clifford algebra of the underlying space.

Our notation and terminology is that of [2]. Unlike the situation treated in [2], we have a plentiful supply of involutions at our disposal. We utilize these involutions to produce a bijection of the planes of V. This bijection is shown to be the same as that induced by arbitrary plane rotations and is used to induce a bijection of the lines of V. The Fundamental Theorem of Projective Geometry is then applied as in [2] to determine the form of the automorphism.

V is a quaternary, anisotropic space over a field F with characteristic F different from 2. If $\sigma, \sigma', \sigma_1, \sigma_2, \ldots, \sigma_1', \sigma_2', \ldots$ are elements of $O(V)$ then $R, R', R_1, R_2, \ldots, R_1', R_2', \ldots$ will be used to denote their respective residual spaces. Since V is anisotropic, all subspaces of V are regular. If R is any subspace of V there is a unique involution σ in $O(V)$ with $R =$ res space σ. Let $\wedge$ be an automorphism of $O^+(V)$. For σ in $O^+(V)$ let σ' denote $\wedge\sigma$.

PROPOSITION 1. *Let σ be a plane rotation that is an involution. Then $\wedge\sigma$ is a plane rotation.*

PROOF. By assumption σ is not in the center of $O^+(V)$. Hence $\wedge\sigma$ is not in the center of $O^+(V)$ and res $\wedge\sigma = 2$. Q.E.D.

Now let R be any plane and let σ be the unique involution in $O^+(V)$ with res space $\sigma = R$. By Proposition 1 we have a bijection of planes $R \mapsto R'$.

PROPOSITION 2. *If R_1 and R_2 are planes with $R_1 \cap R_2$ a line, then $R_1' \cap R_2'$ is a line.*

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PROOF. Let $R_1 \cap R_2 = Fx$. Set $R_i = Fx \perp Fy_i$ and $\sigma_i = \tau_x \tau_{y_i} = \tau_{y_i} \tau_x$, $i = 1, 2$. Then σ_i is an involution with res space $\sigma_i = R_i$. Now $\sigma_1 \sigma_2 = \tau_{y_1} \tau_{y_2}$, a plane rotation. Hence $\sigma_1 \sigma_2$ is not in the center of $O^+(V)$. If $R_1' \cap R_2' = \{0\}$ then $V = R_1' + R_j'$ and $\sigma_1' \sigma_2' = -1_V$ which is in the center of $O^+(V)$. Hence $R_1' \cap R_2'$ is a line. Q.E.D.

PROPOSITION 3. *Let σ be a plane rotation. Then $\wedge \sigma$ is a plane rotation.*

PROOF. We may assume that σ is not an involution by Proposition 1. Express $\sigma = \tau_x \tau_y$ where $R = Fx + Fy$. Choose z in R^* with $z \neq 0$. Let $R_1 = Fx + Fz$ and $R_2 = Fy + Fz$. Let $\sigma_1 = \tau_x \tau_z$ and $\sigma_2 = \tau_z \tau_y$. Then $\sigma_1 \sigma_2 = \tau_x \tau_y = \sigma$ and $\sigma' = \sigma_1' \sigma_2'$. But

$$\text{res space } \sigma' \subseteq \text{res space } \sigma_1' + \text{res space } \sigma_2' = R_1' + R_2'.$$

By Proposition 2, $R_1' + R_2'$ is three dimensional. Since res σ' is even, res σ' is 2. Q.E.D.

PROPOSITION 4. *Let σ_1 and σ_2 be plane rotations with $R_1 = R_2$. Then res space $\sigma_1' =$ res space σ_2'.*

PROOF. The result is clear if $\sigma_1 = \sigma_2^{-1}$ so we assume otherwise. Then σ_1 and σ_2 permute and $\sigma_1 \sigma_2$ is a plane rotation whose residual space is R_1. By Proposition 3, $(\sigma_1 \sigma_2)'$ is a plane rotation. Since $\sigma_1' \sigma_2'$ is a plane rotation and $\sigma_1' \sigma_2' \neq 1$, we have res space $\sigma_1' \cap$ res space $\sigma_2' \neq \{0\}$ by [2, 1.21]. Since σ_1' and σ_2' permute, res space $\sigma_1' =$ res space σ_2'. Q.E.D.

By Propositions 3 and 4 we can produce the same bijection of planes as before if we utilize arbitrary plane rotations.

PROPOSITION 5. *If $\{R_\alpha\}$ is a set of planes and $\bigcap R_\alpha$ is a line, then $\bigcap R_\alpha'$ is a line.*

PROOF. It suffices to show that $R_1' \cap R_2' \cap R_3'$ is a line if R_1, R_2 and R_3 are three distinct planes that intersect in a line. By Propositions 2 and 4, $R_i' \cap R_j'$ is a line if $i \neq j$, $1 \leqslant i, j \leqslant 3$. Let $R_1' \cap R_2' = Fy$, $R_2' \cap R_3' = Fz$ and $R_1' \cap R_3' = Fx$. Choose $\sigma_1' = \tau_x \tau_y$, $\sigma_2' = \tau_y \tau_z$, $\sigma_3' = \tau_z \tau_x$, then σ_i' has res space R_i', $1 \leqslant i \leqslant 3$, and $\sigma_1 \sigma_2 \sigma_3 = 1$. Now express $\sigma_1 = \tau_b \tau_a$, $\sigma_2 = \tau_a \tau_c$, $\sigma_3 = \tau_a \tau_d$ where $R_1 = Fa + Fb$, $R_2 = Fa + Fc$, and $R_3 = Fa + Fd$. Then $\sigma_1 \sigma_2 = \tau_b \tau_c = \sigma_3^{-1} = \tau_d \tau_a$. Hence $Fa \subseteq Fb + Fc$ and $R_1 + R_2 = Fb + Fc$. This is a contradiction. Q.E.D.

Now let l be a line in V. Let $\{R_\alpha\}$ be the set of planes containing l. By Proposition 5, $\bigcap R_\alpha'$ is a line l'. The Fundamental Theorem of Projective Geometry may now be applied to the correspondence $l \mapsto l'$; let g be the semilinear map so obtained.

PROPOSITION 6. *g preserves orthogonality.*

PROOF. Let l_1 and l_2 be orthogonal lines. Choose planes R_1 and R_2 with $l_i \subseteq R_i$, $i = 1,2$, and $(R_1, R_2) = 0$. Choose σ_i noninvolutions with res space $\sigma_i = R_i$ and apply permutability arguments to conclude that R_1' and R_2' are orthogonal planes. Q.E.D.

Now consider the map ϕ_g as in [2]. We have

PROPOSITION 7. *ϕ_g induces an automorphism of $O_4^+(V)$ if and only if g preserves orthogonality.*

PROOF. As in 3.3 of [2]. Note that the assumption $n \geqslant 5$ is used in 3.3 in order to conclude from 2.12 that ϕ_g maps regular plane rotations to regular plane rotations. Our assumption of anisotropy eliminates this need. Q.E.D.

THEOREM 8. *Let $\wedge$ be an automorphism of the group $O^+(V)$ where V is a quaternary anisotropic space defined over a field F of characteristic not two. Then $\wedge$ can be expressed in the form $\wedge = P_\psi \circ \phi_g$ for a unique radial automorphism P_ψ and a unique automorphism of type ϕ_g.*

PROOF. $\wedge$ induces the semilinear map g as above. By Propositions 6 and 7, $\wedge \circ \phi_g^{-1}$ is an automorphism of $O_4^+(V)$. The bijection of planes reduced by $\wedge \circ \phi_g^{-1}$ is identity and hence $\wedge \circ \phi_g^{-1}$ is a radial automorphism P_ψ as in 4.5 of [2]. Uniqueness follows as in [2]. Q.E.D.

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