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BANACH LATTICE STRUCTURES ON SEPARABLE L_p SPACES

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ABSTRACT. A complete characterization of those lattice structures on separable L_p spaces which are Banach lattice structures under the L_p norm is given.

The Banach spaces $L_p(\mu)$ of all (equivalence classes of) μ-measurable real valued functions f such that $|f|^p$ is μ-integrable are among the most important and the most elegant examples of Banach lattices (here we are taking $1 \leqslant p < \infty$ and the norm to be the usual L_p norm given by $\|f\| = (\int |f|^p \, d\mu)^{1/p}$ and the lattice structure to be defined in the usual pointwise manner). In the separable case it is well known that $L_p(\mu)$ is linearly isometric and lattice isomorphic to exactly one of five concrete L_p spaces, namely, $l_p(n)$, l_p, L_p $(= L_p[0, 1])$, $l_p(n) \oplus L_p$, and $l_p \oplus L_p$ where the direct sums are taken in the p sense (see, for example, [3]). It has become of interest to explore the question of classifying the lattice structures, if any, which can be put on a Banach space to make it a Banach lattice (for a Banach lattice we insist that for any element x in the space, $\|x\| = \||x|\|$ where $|x|$ is the modulus of x in the lattice structure). This question is naturally well posed in both the isomorphic theory and the isometric theory of Banach spaces. The isomorphic theory is concerned with when a given Banach space is linearly isomorphic to a Banach lattice and the isometric theory simply replaces 'linearly isomorphic' with 'linearly isometric'.

For example, it is well known that for $p > 1$, a separable space $L_p(\mu)$ admits an unconditional Schauder basis $\{f_n\}$, that is, there is a constant $K \geqslant 1$ such that for any pair of finite sequences $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ of scalars with $|\alpha_i| \leqslant |\beta_i|$ for $i = 1, \ldots, k$, we have that $\|\sum_{i=1}^k \alpha_i f_i\| \leqslant K\|\sum_{i=1}^k \beta_i f_i\|$. The smallest such constant K is called the *unconditional basis constant* for $\{f_n\}$ and it can be easily seen that if we define a new norm on $L_p(\mu)$ by $\|\sum_{n=1}^\infty \alpha_n f_n\|' = \sup\{\|\sum_{n=1}^\infty \beta_n \alpha_n f_n\| : |\beta_n| \leqslant 1\}$, then this norm is equivalent to the original norm and the f_n's have unconditional basis constant equal to one with respect to this new norm. Moreover, it follows readily that under the new norm that $L_p(\mu)$ can be given the structure of a purely atomic Banach lattice simply by defining $\sum_{n=1}^\infty \alpha_n f_n \geqslant 0$ if and only if $\alpha_n \geqslant 0$ for all n (by purely atomic we mean that every positive element dominates some atom, e.g., $\sum_{n=1}^\infty \alpha_n f_n \geqslant \alpha_k f_k$ where $\alpha_k > 0$). Thus $L_p(\mu)$ in this new norm and new lattice structure is not lattice isomorphic to $L_p(\mu)$ in the original norm and lattice structure (however, the two are linearly isomorphic).

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Recently the second named author and Abramovič [1] have studied Banach lattices which are linearly isomorphic to $L_p(\mu)$. For example, they proved that l_p is not linearly isomorphic to an atomless Banach lattice and that L_1 and L_2 have the property that if they are linearly isomorphic to an atomless Banach lattice, then there is also a linear isomorphism which preserves the lattice structure, i.e., the lattice structures on L_1 and L_2 are unique up to order isomorphism.

In this paper we consider only the Banach lattices which are linearly isometric to a separable $L_p(\mu)$. That is, we assume that $L_p(\mu)$ has some lattice structure such that it is a Banach lattice under the L_p-norm. We are able to give a complete characterization of such lattice structures and show that $L_p(\mu)$ is order isomorphic (but not order isometric necessarily) in its natural order to each of these structures. One consequence of this is the known result that a monotone basis for $L_p[0, 1]$ $(1 < p < \infty)$ cannot have unconditional basis constant equal to one (see [2] for a characterization of monotone bases in $L_p[0, 1]$).

To see how to obtain some Banach lattice structures on $L_p(\mu)$ let us consider the two-dimensional space $l_p(2)$ $(1 \leqslant p < \infty,\ p \neq 2)$. Let e_1, e_2 denote the standard unit vector basis in $l_p(2)$. Then $f_1 = (e_1 + e_2)/2^{1/p}$ and $f_2 = (e_1 - e_2)/2^{1/p}$ together form an unconditional basis for $l_p(2)$ with basis constant 1. Moreover, $\|\varepsilon_1 f_1 + \varepsilon_2 f_2\|_p = \|f_1 + f_2\|_p$ where $\varepsilon_i \varepsilon \{1, -1\}$ for $i = 1$, 2. Thus it follows readily that the cone spanned by f_1, f_2 (i.e., $C = \{af_1 + bf_2$: $a, b \geqslant 0\}$) is a lattice cone and the p norm is a Banach lattice norm for this lattice. We shall denote this space by $E_p(2)$. Thus we have that l_p is linearly isometric to $(\bigoplus\sum_{n=1}^{\infty} F_n)_p$ where for each n, $F_n = l_p(2)$ or $F_n = E_p(2)$. Clearly if at least one $F_n = E_p(2)$, then $(\bigoplus\sum_{n=1}^{\infty} F_n)_p$ is not an L_p-space as a Banach lattice, since, for example, the norm is not p-additive (see [3]). Thus it is a different Banach lattice structure on l_p. More generally, if ν is any measure and X is a Banach lattice, then $L_p(\nu, X)$ is the Banach lattice of all measurable X-valued functions f such that $\int \|f(t)\|^p \, d\nu(t) = \|f\|_p^p$ is finite. Moreover, if X is an L_p space, then $L_p(\nu, X)$ is also an L_p space since the norm in $L_p(\nu, X)$ is p-additive (see [3]). Since $E_p(2)$ is linearly isometric to $l_p(2)$, for any measure μ, $L_p(\mu, E_p(2))$ is linearly isometric to $L_p(\mu, l_p(2))$ which, in turn, is linearly isometric and order isomorphic to $L_p(\mu) \oplus_p L_p(\mu)$. Thus, for any measure ν, $L_p(\nu) \oplus_p L_p(\mu, E_p(2))$ is linearly isometric to $L_p(\nu) \oplus_p L_p(\mu) \oplus_p L(\mu)$, but not order isometric to it.

We shall prove that if X is a separable Banach lattice which is linearly isometric to a separable space $L_p(\mu)$, then there are measures ν_1 and ν_2 (possibly one of them is zero) such that X is linearly isometric and order isomorphic to $L_p(\nu_1) \oplus_p L_p(\nu_2, E_p(2))$.

We shall need some terminology and elementary results from the theory of Banach lattices. The norm on a Banach lattice X is said to be *order continuous* if whenever Z is a downwards directed set of positive elements of X with infimum equal to zero, the infimum of the norms of the elements of Z is also zero. A *band* in a Banach lattice X is a closed ideal B such that if $Z \subset B$ and the supremum of Z exist in X, then the supremum is in B. It is known, for example, that if the norm is order continuous in X, then each net in X which has a supremum converges to this supremum in norm. Thus, any closed ideal is automatically a band (see [3]). Moreover, if the norm is order continuous, then for any band B there is a complementary band $B^\perp = \{x \in X: |x| \wedge |y| = 0 \text{ for all } y \in B\}$ such that $X = B \oplus B^\perp$. The natural projection of X onto B with kernel $B^\perp$ has norm one and is called a *band projection*. The first lemma has an easy proof and we omit it.

LEMMA 1. *Let X be a Banach lattice with order continuous norm. If $Y \subset X$ is a closed subspace and $P(Y) \subset Y$ for all band projections P on X, then Y is a band in X.*

LEMMA 2. *Let X be a Banach lattice. If P is a band projection X, then $P = \frac{1}{2}(I + V)$ where V is a linear isometry and $V^2 = I$.*

PROOF. Let $Q = I - P$, the complementary band projection to P. Clearly if such a V exists, it must be $V = 2P - I$. Thus we only show that $2P - I$ is an isometry (clearly $(2P - I)^2 = I$). Let $x \in X$. Then

$$\begin{aligned} V(x) &= (2P - I)(x^+ - x^-) = 2P(x^+) - 2P(x^-) - x^+ + x^- \\ &= 2P(x^+) - 2P(x^-) - P(x^+) - (I - P)(x^+) \\ &\quad + P(x^-) + (I - P)(x^-) \\ &= P(x^+) - P(x^-) - (I - P)(x^+) + (I - P)(x^-) \end{aligned}$$

and the terms are pairwise disjoint. Thus

$$\begin{aligned} \|V(x)\| &= \|P(x^+) + (I - P)(x^+) - P(x^-) - (I - P)(x^-)\| \\ &= \|x^+ - x^-\| = \|x\|. \end{aligned}$$

LEMMA 3. *Let X and Y be Banach lattices with order continuous norm. If φ is a linear isometry from X onto Y such that for each band $M \subset X$, $\varphi(M)$ is a band in Y and for each band $N \subset Y$, $\varphi^{-1}(N)$ is a band in X, then there is an order preserving linear isometry of X onto Y.*

PROOF. Let $x \in X$. Then $\varphi(x^{\perp\perp})$ is a band in Y containing $\varphi(x)$. Thus $\varphi(x^{\perp\perp}) \supset \varphi(x)^{\perp\perp}$. Since φ^{-1} has the same property, $\varphi^{-1}(\varphi(x)^{\perp\perp}) \supset (\varphi^{-1}\varphi(x))^{\perp\perp} = x^{\perp\perp}$. Thus $\varphi(x)^{\perp\perp} \supset \varphi(x^{\perp\perp})$ and we obtain $\varphi(x^{\perp\perp}) = \varphi(x)^{\perp\perp}$. If $|x| \wedge |y| = 0$, then $x^{\perp\perp} \cap y^{\perp\perp} = \{0\}$. Hence $\varphi(x)^{\perp\perp} \cap \varphi(y)^{\perp\perp} = \{0\}$ and $|\varphi(x)| \wedge |\varphi(y)| = 0$.

Now let $C = \{x \in X^+: \varphi(x) \geqslant 0\}$. Clearly C is a closed cone in X and $V = C - C$ is a closed linear subspace of X. Let P be a band projection on X and $x \in C$. Then $|\varphi(Px)| \wedge |\varphi(x - Px)| = 0$ and $0 \leqslant \varphi(x) = \varphi(Px) + \varphi(x - Px)$. Thus $\varphi(Px) \geqslant 0 \Rightarrow Px \in C$. Hence $P(V) \subset V$ and by Lemma 1, V is a band in X. Now suppose $x \in V^\perp$ and $x \geqslant 0$. Then $\varphi(x) \leqslant 0$. For, suppose $\varphi(x) = y_1 - y_2$ with $y_1 > 0$, $y_2 \geqslant 0$, $y_1 \wedge y_2 = 0$. Then $x = \varphi^{-1}(y_1) - \varphi^{-1}(y_2)$ implies $\varphi^{-1}(y_1) \geqslant 0$ and $\varphi^{-1}(y_1) \in V$. Thus we define ψ by $\psi | V = \varphi$ and $\psi | V^\perp = -\varphi$. Then ψ is a linear isometry of X onto X order preserving.

The main basis of our analysis is the theorem of Lamperti [5] on the representation of linear isometries on $L_p(\mu)$. Since we are assuming that $L_p(\mu)$ is separable, we can take $L_p(\mu) = L_p(T, \Sigma, \mu)$ where $T = T_1 \cup T_2$ where either $T_1 = \varnothing$ or $T_1 = [0, 1]$, $T_2 = \varnothing$ or T_2 is a countable set in [2,3] (for example). Moreover, $\Sigma | T_1$ can be taken to be the Lebesgue measurable sets on T_1 (if $T \neq \varnothing$) and $\Sigma|T_2$ to be the set of all subsets of T_2 (if $T_2 \neq \varnothing$), and μ can be taken to be Lebesgue measure on $\Sigma|T_1$ (if $T_1 \neq \varnothing$), counting measure on $\Sigma|T_2$ if T_2 is finite and nonempty, and if $T_2 = \{t_n\}$, we take $\mu(t_n) = 1/2^n$ (recall the representation theory for separable $L_p(\mu)$ spaces).

THEOREM (LAMPERTI). *Let $1 \leqslant p < \infty$ and $p \neq 2$ and V be a linear isometry of $L_p(\mu)$ onto $L_p(\mu)$. Then there is a measurable function h on T and a measurable $\varphi\colon T \to T$ such that $\varphi(T_1) = T_1$ (a.e.), $\varphi(T_2) = T_2$, φ is essentially one-to-one (i.e., one-to-one except on a set of measure zero), φ^{-1} is measurable, and $Vf = h(f \circ \varphi)$ for all $f \in L_p(\mu)$ and $\int_{\varphi^{-1}(E)}|h|^p\,d\mu = \mu(E)$ for all measurable sets $E \subset T$, and $|h| = 1$ on T_2.*

If $V^2 = I$ (the identity), then it is easy to see that $\varphi^2 =$ identity and that $h = 1/h \circ \varphi$ since $1 = V^2(1)$.

Moreover, we can decompose T into three disjoint measurable sets A_0, A_1, A_2 where $A_0 = \{t\colon \varphi(t) = t\}$, φ maps A_1 onto A_2 and A_2 onto A_1, and $|h| = 1$ on A_0, e.g., put $A_1 = \{t\colon \varphi(t) > t\}$ and $A_2 = \{t\colon \varphi(t) < t\}$.

The crucial part of our proof depends on an analysis via the Lamperti theorem of the decomposition of T whenever we have two projections $P_1 = \frac{1}{2}(I + V_1)$ and $P_2 = \frac{1}{2}(I + V_2)$ where V_1 and V_2 are idempotent linear isometries on $L_p(\mu)$ and P_1 dominates P_2, i.e., $P_1P_2 = P_2P_1 = P_2$. From the Lamperti theorem we obtain functions h_1, h_2, φ_1, φ_2 with the properties listed in the theorem and the following remark. Let A_0, A_1, A_2 be the above described sets relative to φ_1. Then on A_0 we have that $h_1^2 = 1$. Thus $A_0 = A_0^+ \cup A_0^-$ where $A_0^+ = \{t \in A_0\colon h_1(t) = 1\}$ and $A_0^- = \{t\colon h_1(t) = -1\}$ and φ_1 maps A_1 onto A_2 and A_2 onto A_1. Now

$$P_1P_2 = \tfrac{1}{4}\left[I + V_1 + V_2 + V_1V_2\right] = \tfrac{1}{4}\left[I + V_1 + V_2 + V_2V_1\right] = P_2P_1$$

so that $V_1V_2 = V_2V_1$. Since

$$V_1V_2f = h_1 \cdot h_2 \circ \varphi_1 \cdot f \circ \varphi_2\varphi_1 = h_2 \cdot h_1 \circ \varphi_2 \cdot f \circ \varphi_1 \circ \varphi_2$$

for all $f \in L_p(\mu)$, by putting $f = 1$ we obtain that $h_1 \cdot h_2 \circ \varphi_1 = h_2 \cdot h_1 \circ \varphi_2$ and, thus, $f \circ \varphi_2 \circ \varphi_1 = f \circ \varphi_1 \circ \varphi_2$ for all $f \in L_p(\mu)$. Thus by taking f to be the identity function on T (recall that we conveniently choose $T \subset [0, 3]$), we see that $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$. Thus, if $t \in A_0$, then $\varphi_1(\varphi_2(t)) = \varphi_2(\varphi_1(t)) = \varphi_2(t)$ and we have that $\varphi_2(A_0) \subset A_0$. On the other hand, if $\varphi_2(\varphi_1(t)) = \varphi_1(\varphi_2(t)) = \varphi_2(t)$, then $t = \varphi_1(t)$ and it follows that $\varphi_2(A_1 \cup A_2) \subset A_1 \cup A_2$. Since φ_2 is onto (a.e.), it follows that $\varphi_2(A_0) = A_0$ and $\varphi_2(A_1 \cup A_2) = A_1 \cup A_2$ (a.e.).

We shall now show that A_0, A_1, A_2 can be further decomposed by using the fact that $P_1P_2 = P_2P_1 = P_2$.

LEMMA 4. (A) *Let $C = \{t \in A_1 \cup A_2\colon \varphi_2(t) = t\}$. Then for*

$$D = (A_1 \cup A_2) \setminus C$$

we have the following relations (a.e.).

(i) $h_2 = -1$ *on* C,

(ii) $\varphi_1 = \varphi_2$ *on* D,

(iii) $h_1 = h_2$ *on* $D \cup A_0^-$,

(iv) φ_2 *is the identity on* A_0^-.

(B) *Let* $B_0 = \{t \in A_0^+ : \varphi_2(t) = t\}$, $B_1 = \{t \in A_0^+ : \varphi_2(t) > t\}$ *and* $B_2 = \{t \in A_0^+ : \varphi_2(t) < t\}$. *Then* $A_0^+ = B_0 \cup B_1 \cup B_2$, $\varphi_2(B_1) = B_2$, $\varphi_2(B_2) = B_1$.

PROOF. Since φ_1 maps A_1 onto A_2 and A_2 onto A_1, and $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, it follows that $\varphi_1(C \cap A_1) = C \cap A_2$. Let $f = \mathfrak{X}_{C \cap A_1}$, the characteristic function of $C \cap A_1$. Then $P_2 f = \frac{1}{2}(f + h_2 f)$ $(f \circ \varphi_2 = f)$ has its support contained in $C \cap A_1$ and

$$P_1 P_2 f = \tfrac{1}{4}\left[f + h_2 f + h_1(f \circ \varphi_1 + h_2 \circ \varphi_1 \cdot f \circ \varphi_1)\right] = P_2 f.$$

Thus it follows that $P_2 f = 0$ since $h_1(f \circ \varphi_1 + h_2 \circ \varphi_1 \cdot f \circ \varphi_1) = 0$ on $C \cap A_1$. A similar argument shows that $P_2 g = 0$ if $g = \mathfrak{X}_{C \cap A_1}$. Thus $P_2(f + g) = \frac{1}{2}(f + g + h_2(f + g)) = 0$ and it follows that $h_2 = -1$ on C. Hence part (i) of (A) is true.

Suppose that $\varphi_1|D \neq \varphi_2|D$. Then there is a subset Z of D with positive measure such that Z, $\varphi_1(Z)$, $\varphi_2(Z)$ are pairwise disjoint (consider $Z = \{t \in A_1 \cap D: \varphi_1(t) > \varphi_2(t)$ and $\varphi_2(t) > t\}$ or the corresponding sets with A_1 replaced by A_2 and/or greater than by less than). Let $f = \mathfrak{X}_Z$. Then $\operatorname{supp} P_2 f = Z \cup \varphi_2(Z)$ but $\operatorname{supp} P_1 P_2 f \supset \varphi_1(Z)$ so that $P_2 f \neq P_1 P_2 f$. Thus part (ii) of (A) is established.

Since $\varphi_1 = \varphi_2$ on D, $\varphi_1^{-1} = \varphi_1$, and $\varphi_2^{-1} = \varphi$, from the Lamperti theorem it follows that $|h_1| = |h_2|$ on $\varphi_1(D) = \varphi_2(D)$ and since $h_1 h_1 \circ \varphi_1 = h_2 h_2 \circ \varphi_2 = 1$, it follows that $|h_1| = |h_2|$ on D. Now suppose that $h_1 = -h_2$ on a subset Z of D with positive measure and put $f = \mathfrak{X}_{Z \cap A_1}$. Then $P_2 f = \frac{1}{2}(f + h_2 \mathfrak{X}_{\varphi_2(Z \cap A_1)})$,

$$\begin{aligned} P_1 P_2 f &= \tfrac{1}{4}\left[f + h_2 \mathfrak{X}_{\varphi_2(Z \cap A_1)} + h_1 f \circ \varphi_1 + h_1(h_2 \circ \varphi_2)\mathfrak{X}_{\varphi_1\varphi_2(Z \cap A_1)}\right] \\ &= \tfrac{1}{4}\left(f + h_2 \mathfrak{X}_{\varphi_2(Z \cap A_1)} + h_1 \mathfrak{X}_{\varphi_1(Z \cap A_1)} + h_1 h_2 \circ \varphi_2 f\right) \\ &= \tfrac{1}{4}\left(f + h_2 \mathfrak{X}_{\varphi_2(Z \cap A_1)} - h_2 \mathfrak{X}_{\varphi_2(Z \cap A_1)} - f\right) = 0 \end{aligned}$$

since $h_1 = -h_2$ on $\varphi_2(Z \cap A_1)$. A similar result holds for $f = \mathfrak{X}_{Z \cap A_2}$. Thus we obtain a contradiction to the assumption that $P_1 P_2 = P_2$ since one of $Z \cap A_1$ and $Z \cap A_2$ has positive measure. Hence $h_1 = h_2$ on D.

If $f \in L_p(\mu)$ and $\operatorname{supp} f \subset A_0^-$, then $P_1 f = \frac{1}{2}(f + h_1 f \circ \varphi_1) = \frac{1}{2}(f - f) = 0$. Thus $P_2 f = P_2 P_1 f = 0$. Since $P_2 f = \frac{1}{2}(f + h_2 f \circ \varphi_2)$, it follows by putting $f = \mathfrak{X}_{A_0^-}$ that $h_2 = -1$ and φ_2 is the identity on A_0^-. Thus part (iv) of (A) is valid.

Part (B) follows from the observation that $\varphi_2(A_0) = A_0$ since $\varphi_1\varphi_2 = \varphi_2\varphi_1$ and that φ_2 is the identity on A_0^- so that $\varphi_2(A_0^+) = A_0^+$.

We shall continue the above ordering on projections in the next lemma. Specifically, if we have a sequence of band projections P_n on a Banach lattice X with order continuous norm and if $P_n P_{n+1} = P_{n+1} P_n = P_{n+1}$ for all n, then $\inf P_n = P$ is also a band projection, where $P(x) = \lim_n P_n(x)$ for all $x \in X$. Note that since the norm is order continuous, if $x \geqslant 0$, $\{P_n(x)\}$ is a decreasing sequence of positive elements and, hence, must converge in norm to its infimum. Thus P can be so defined. It is easy to check that P is a projection onto $\bigcap_{n=1}^{\infty} P_n(X) = M$ and that M is a band in X. To see that P is the band projection note that for any $x \geqslant 0$, $Px \wedge (x - Px) = \lim_n P_n x \wedge (x - P_n x)$.

We assume that the Banach space $L_p(\mu)$ has a lattice structure which makes it a Banach lattice in the L_p-norm for the next lemma.

LEMMA 5. *There is a band projection P which is minimal with respect to property*:

($*$) *For each set $E \subset T$ of positive measure there is an $f \in L_p(\mu)$ such that $(Pf)|E \neq 0$.*

PROOF. Since $L_p(\mu)$ is separable it suffices to show that if $P_1 \geqslant P_2 \geqslant \cdots$ is a sequence of band projections in the assumed Banach lattice structure satisfying property ($*$), then so does $P = \inf_n P_n$ (the norm is order continuous on the Banach lattice structure under consideration since as a Banach space $L_p(\mu)$ does not contain a linearly isomorphic copy of c_0; see [8]).

Let $P_n = \frac{1}{2}(I + V_n)$ where V_n is an idempotent linear isometry on $L_p(\mu)$ and $V_n f = h_n(f \circ \varphi_n)$ where h_n, φ_n are given by Lamperti's theorem. The condition ($*$) is equivalent to $h_n \neq 1$ (a.e.) on sets where φ_n is the identity since $(Pf) \cdot \mathcal{X}_E = \frac{1}{2}(f\mathcal{X}_E + h \cdot (f \circ \varphi) \cdot \mathcal{X}_E)$. For, if $h = -1$ on E and φ is the identity on E, then $(Pf) \cdot \mathcal{X}_E = 0$ for all f and if $(Pf) \cdot \mathcal{X}_E = 0$ for all f, then for $f = \mathcal{X}_F$, $\mathcal{X}_F = -h\mathcal{X}_{\varphi(f)}$ for $F \subset E$, which implies that $h = -1$ on E and φ is the identity on E.

Let A_0, n, $A_{1,n}$, $A_{2,n}$ be the sets associated with P_n in the remarks preceding Lemma 4. Then using Lemma 4 we can define h and φ by $h(t) = h_n(t)$ and $\varphi(t) = \varphi_n(t)$ if $t \in \lim\sup(A_{1,k} \cup A_{2,k})$ and n is the least integer such that $t \in A_{1,k} \cup A_{2,k}$ and $h(t) = 1$ and $\varphi(t) = t$ if $t \in \lim\inf A_{0,k}$.

Since $\|h_n\| = 1$, $\|h\| = 1$ and $Vf = h(f \circ \varphi) = \lim_n V_n f$ is an isometry. Moreover, $P = \frac{1}{2}(I + V)$, and since $h \neq -1$ where $\varphi =$ identity, P satisfies property ($*$).

We now come to the main theorem of the paper.

THEOREM. *Let X be a Banach lattice linearly isometric to $L_p(\mu)$. Then there are measurable sets A_0, A_1 in T such that X is linearly order isometric to $L_p(A_0) \oplus_p L_p(A_1, E_p(2))$.*

PROOF. Let P be any band projection in X and A_0, A_1, A_2 the sets associated with P (see Lemma 4 and the remarks preceding it). Then $L_p(A_1 \cup A_2)$ and $L_p(A_0)$ are bands in X and in each case the band projections are given by multiplication by the associated characteristic functions. We prove this for $L_p(A_1 \cup A_2)$ and the other proof is similar. Suppose Q is a band projection and $Q \leqslant P$. Then by Lemma 4 it follows that the support of $Q(f \cdot \mathcal{X}_{A_1 \cup A_2})$ is contained in $A_1 \cup A_2$ for all $f \in L_p(\mu)$. If $Q \leqslant I - P$, then Lemma 4 shows the same result. Thus by Lemma 1, $L_p(A_1 \cup A_2)$ is a band in X. Let Q be the band projection of X onto $L_p(\mu)$. Let h_2, φ_2 be the functions such that $Qf = ½(f + h_2 f \circ \varphi_2)$ for all $f \in L_p(\mu)$. Then for any measurable set $Z \subset A_1 \cup A_2$.

$$\mathcal{X}_Z = Q(\mathcal{X}_Z) = \tfrac{1}{2}(\mathcal{X}_Z + h_2\mathcal{X}_{\varphi_2(Z)}).$$

Thus $h_2 = 1$ on Z and $\varphi_2(Z) = Z$. By Lemma 4 we conclude that φ_2 is the identity on $A_1 \cup A_2$. Now for $f \in L_p(\mu)$,

$$Qf = \tfrac{1}{2}(f + h_2 f \circ \varphi_2) \cdot \mathcal{X}_{A_1 \cup A_2} = f \cdot \mathcal{X}_{A_1 \cup A_2}.$$

We now assume that P is minimal with respect to property ($*$). Let $Q \leqslant P$ and let B_0, B_1, B_2 be the sets given by Lemma 4 for Q. Then $\mu(B_1 \cup B_2) = 0$. For, if not, then by the above $L_p(B_1 \cup B_2)$ is a band in the range of P and $Q_1 f = Pf - f \cdot \mathcal{X}_{B_1 \cup B_2}$ defines a band projection with property ($*$) which is smaller than P. We define u: $L_p(\mu) \to L_p(A_0) \oplus_p L_p(A_1, E_p(2))$ as follows:

$$u(f) = f \cdot \mathcal{X}_{A_0} + \left[(Pf) \cdot \mathcal{X}_{A_1}(e_1 + e_2) + (f - Pf) \cdot \mathcal{X}_{A_1}(e_1 - e_2)\right].$$

A routine calculation shows that u is a linear isometry and onto. By Lemma 3 we only need to show that u and u^{-1} both carry bands onto bands. If Q is a band projection and $Q \leqslant P$, then let B_0, and D be the sets in Lemma 4 for Q. Then for $f \in L_p(\mu)$,

$$(Qf) \cdot \mathcal{X}_{A_0} = f \cdot \mathcal{X}_{B_0^+}$$

and

$$(Qf) \cdot \mathcal{X}_{A_1} = \tfrac{1}{2}\left(f \cdot \mathcal{X}_{A_1 \cap D} + (h_1 f \circ \varphi_1) \cdot \mathcal{X}_{A_1 \cap D}\right)$$

where h_1, φ_1 are the functions associated with P. That is,

$$Q(X) = L_p(B_0^+) \oplus_p \left[P(X) \cap L_p(D)\right]$$

and

$$u(Q(X)) = L_p(B_0^+) \oplus_p L_p(A_1 \cap D)(e_1 + e_2).$$

If $Q \leqslant I - P$, then $I - Q \leqslant P$ and if B_0 and D are the sets for $I - Q$ as above, then a similar argument shows that

$$Q(X) = L_p(B_0^-) \oplus_p \left[(I - P)(X) \cap L_p(D)\right]$$

and $u(Q(X)) = L_p(B_0^-) \oplus_p L_p(D)(e_1 - e_2)$.

If $(g_1, g_2, g_3) \in L_p(A_0) \oplus_p L_p(A_1, E_p(2))$, then $f = g \cdot \mathcal{X}_{A_0} + (g_2 + g_3) \cdot \mathcal{X}_{A_1} + [h_1(g_2 - g_3) \circ \varphi_1] \cdot \mathcal{X}_{A_2}$, $u(f) = g$ and proofs similar to the above show that u^{-1} carries bands onto bands.

Added in proof. The first author and S. Bernau have recently shown that the same theorem holds in the general case. This was done by characterizing bicontractive projections on $L_p(\mu)$ as those projections such that $2P - I$ is an isometry.

References

1. Ju. A. Abramovič and P. Wojtaszczyk, *On the uniqueness of order in spaces l_p and $L_p[0, 1]$ $(1 \leqslant p \leqslant \infty)$* (Russian) (to appear).

2. L. E. Dor and E. Odell, *Monotone bases in L_p* (to appear).

3. H. Elton Lacey, *The isometric theory of classical Banach spaces*, Die Grundlehren der math. Wissenschaften, Band 208, Springer-Verlag, Berlin and New York, 1974.

4. E. Lacey and P. Wojtaszczyk, *Non-atomic M spaces can have l_1 as a dual space* (to appear).

5. J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math. **8**(1958), 459–466. MR **21** #3764.

6. J. Lindenstrauss and A. Pelczyński, *Contributions to the theory of classical Banach spaces*, J. Functional Analysis **8**(1971), 225–249. MR **45** #863.

7. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, Lecture Notes in Math., vol. 388, Springer-Verlag, Berlin and New York, 1973.

8. P. Meyer-Nieberg, *Charakterisierung einiger topologischer und ordnungstheoretischer Eigenschaften von Banachverbanden mit Hilfe disjunkter Folgen*, Arch Math. **24**(1973), 640–647.

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