BANACH LATTICE STRUCTURES ON SEPARABLE $L_p$ SPACES

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Abstract. A complete characterization of those lattice structures on separable $L_p$ spaces which are Banach lattice structures under the $L_p$ norm is given.

The Banach spaces $L_p(\mu)$ of all (equivalence classes of) $\mu$-measurable real valued functions $f$ such that $|f|^p$ is $\mu$-integrable are among the most important and the most elegant examples of Banach lattices (here we are taking $1 \leq p < \infty$ and the norm to be the usual $L_p$ norm given by $\|f\| = (\int |f|^p \, d\mu)^{1/p}$ and the lattice structure to be defined in the usual pointwise manner). In the separable case it is well known that $L_p(\mu)$ is linearly isometric and lattice isomorphic to exactly one of five concrete $L_p$ spaces, namely, $l_p(\mathbb{N})$, $l_p$, $L_p(\mathbb{N})$, $l_p(\mathbb{N}) \oplus L_p$, and $l_p \oplus L_p$ where the direct sums are taken in the $p$ sense (see, for example, [3]). It has become of interest to explore the question of classifying the lattice structures, if any, which can be put on a Banach space to make it a Banach lattice (for a Banach lattice we insist that for any element $x$ in the space, $\|x\| = \||x||$ where $|x|$ is the modulus of $x$ in the lattice structure). This question is naturally well posed in both the isomorphic theory and the isometric theory of Banach spaces. The isomorphic theory is concerned with when a given Banach space is linearly isomorphic to a Banach lattice and the isometric theory simply replaces ‘linearly isomorphic’ with ‘linearly isometric’.

For example, it is well known that for $p > 1$, a separable space $L_p(\mu)$ admits an unconditional Schauder basis $(f_n)$, that is, there is a constant $K > 1$ such that for any pair of finite sequences $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ of scalars with $\|\alpha_i\| < \|\beta_i\|$ for $i = 1, \ldots, k$, we have that $\|\sum_{i=1}^k \alpha_i f_i\| \leq K \|\sum_{i=1}^k \beta_i f_i\|$. The smallest such constant $K$ is called the unconditional basis constant for $(f_n)$ and it can be easily seen that if we define a new norm on $L_p(\mu)$ by $\|\sum_{n=1}^\infty \alpha_n f_n\|'' = \sup \{\|\sum_{n=1}^\infty \beta_n \alpha_n f_n\| : \|\beta_n\| < 1\}$, then this norm is equivalent to the original norm and the $f_n$'s have unconditional basis constant equal to one with respect to this new norm. Moreover, it follows readily that under the new norm that $L_p(\mu)$ can be given the structure of a purely atomic Banach lattice simply by defining $\sum_{n=1}^\infty \alpha_n f_n > 0$ if and only if $\alpha_n > 0$ for all $n$ (by purely atomic we mean that every positive element dominates some atom, e.g., $\sum_{n=1}^\infty \alpha_n f_n > \alpha_k f_k$ where $\alpha_k > 0$). Thus $L_p(\mu)$ in this new norm and new lattice structure is not lattice isomorphic to $L_p(\mu)$ in the original norm and lattice structure (however, the two are linearly isomorphic).

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Recently the second named author and Abramović [1] have studied Banach lattices which are linearly isomorphic to $L_p(\mu)$. For example, they proved that $l_p$ is not linearly isomorphic to an atomless Banach lattice and that $L_1$ and $L_2$ have the property that if they are linearly isomorphic to an atomless Banach lattice, then there is also a linear isomorphism which preserves the lattice structure, i.e., the lattice structures on $L_1$ and $L_2$ are unique up to order isomorphism.

In this paper we consider only the Banach lattices which are linearly isometric to a separable $L_p(\mu)$. That is, we assume that $L_p(\mu)$ has some lattice structure such that it is a Banach lattice under the $L_p$-norm. We are able to give a complete characterization of such lattice structures and show that $L_p(\mu)$ is order isomorphic (but not order isometric necessarily) in its natural order to each of these structures. One consequence of this is the known result that a monotone basis for $L_p[0, 1]$ ($1 < p < \infty$) cannot have unconditional basis constant equal to one (see [2] for a characterization of monotone bases in $L_p[0, 1]$).

To see how to obtain some Banach lattice structures on $L_p(\mu)$ let us consider the two-dimensional space $l_p(2)$ ($1 < p < \infty$, $p \neq 2$). Let $e_1$, $e_2$ denote the standard unit vector basis in $l_p(2)$. Then $f_1 = (e_1 + e_2)/2^{1/p}$ and $f_2 = (e_1 - e_2)/2^{1/p}$ together form an unconditional basis for $l_p(2)$ with basis constant 1. Moreover, $\|\varepsilon_i f_1 + \varepsilon_2 f_2\|_p = \|f_1 + f_2\|_p$ where $\varepsilon_i \in \{-1, 1\}$ for $i = 1, 2$. Thus it follows readily that the cone spanned by $f_1, f_2$ (i.e., $C = \{af_1 + bf_2 : a, b > 0\}$) is a lattice cone and the $p$ norm is a Banach lattice norm for this lattice. We shall denote this space by $E_p(2)$. Thus we have that $l_p$ is linearly isometric to $(\bigoplus_{n=1}^{\infty} F_n)_p$ where for each $n$, $F_n = l_p(2)$ or $F_n = E_p(2)$. Clearly if at least one $F_n = E_p(2)$, then $(\bigoplus_{n=1}^{\infty} F_n)_p$ is not an $L_p$-space as a Banach lattice, since, for example, the norm is not $p$-additive (see [3]). Thus it is a different Banach lattice structure on $l_p$. More generally, if $\nu$ is any measure and $X$ is a Banach lattice, then $L_p(\nu, X)$ is the Banach lattice of all measurable $X$-valued functions $f$ such that $\int \|f(t)\|^p d\nu(t) = \|f\|^p_\nu$ is finite. Moreover, if $X$ is an $L_p$ space, then $L_p(\nu, X)$ is also an $L_p$ space since the norm in $L_p(\nu, X)$ is $p$-additive (see [3]). Since $E_p(2)$ is linearly isometric to $l_p(2)$, for any measure $\mu$, $L_p(\mu, E_p(2))$ is linearly isometric to $L_p(\mu, l_p(2))$ which, in turn, is linearly isometric and order isomorphic to $L_p(\mu, E_p(2))$. Thus, for any measure $\nu$, $L_p(\nu) \oplus L_p(\mu, E_p(2))$ is linearly isometric to $L_p(\nu) \oplus L_p(\mu) \oplus L_p(\mu)$, but not order isometric to it.

We shall prove that if $X$ is a separable Banach lattice which is linearly isometric to a separable space $L_p(\mu)$, then there are measures $\nu_1$ and $\nu_2$ (possibly one of them is zero) such that $X$ is linearly isometric and order isomorphic to $L_p(\nu_1) \oplus L_p(\nu_2, E_p(2))$.

We shall need some terminology and elementary results from the theory of Banach lattices. The norm on a Banach lattice $X$ is said to be order continuous if whenever $Z$ is a downwards directed set of positive elements of $X$ with infimum equal to zero, the infimum of the norms of the elements of $Z$ is also zero. A band in a Banach lattice $X$ is a closed ideal $B$ such that if $Z \subset B$ and the supremum of $Z$ exist in $X$, then the supremum is in $B$. It is known, for example, that if the norm is order continuous in $X$, then each net in $X$ which has a supremum converges to this supremum in norm. Thus, any closed ideal
is automatically a band (see [3]). Moreover, if the norm is order continuous, then for any band $B$ there is a complementary band $B^\perp = \{ x \in X : |x| \perp |y| = 0 \text{ for all } y \in B \}$ such that $X = B \oplus B^\perp$. The natural projection of $X$ onto $B$ with kernel $B^\perp$ has norm one and is called a band projection. The first lemma has an easy proof and we omit it.

**Lemma 1.** Let $X$ be a Banach lattice with order continuous norm. If $Y \subset X$ is a closed subspace and $P(Y) \subset Y$ for all band projections $P$ on $X$, then $Y$ is a band in $X$.

**Lemma 2.** Let $X$ be a Banach lattice. If $P$ is a band projection $X$, then $P = \frac{1}{2}(I + V)$ where $V$ is a linear isometry and $V^2 = I$.

**Proof.** Let $Q = I - P$, the complementary band projection to $P$. Clearly if such a $V$ exists, it must be $V = 2P - I$. Thus we only show that $2P - I$ is an isometry (clearly $(2P - I)^2 = I$). Let $x \in X$. Then
\[
V(x) = (2P - I)(x^+ - x^-) = 2P(x^+) - 2P(x^-) - x^+ + x^- \\
= 2P(x^+) - 2P(x^-) - P(x^+) - (I - P)(x^+) \\
+ P(x^-) + (I - P)(x^-) \\
= P(x^+) - P(x^-) - (I - P)(x^+) + (I - P)(x^-)
\]
and the terms are pairwise disjoint. Thus
\[
\|V(x)\| = \|P(x^+) + (I - P)(x^+) - P(x^-) - (I - P)(x^-)\| \\
= \|x^+ - x^-\| = \|x\|.
\]

**Lemma 3.** Let $X$ and $Y$ be Banach lattices with order continuous norm. If $\varphi$ is a linear isometry from $X$ onto $Y$ such that for each band $M \subset X$, $\varphi(M)$ is a band in $Y$ and for each band $N \subset Y$, $\varphi^{-1}(N)$ is a band in $X$, then there is an order preserving linear isometry of $X$ onto $Y$.

**Proof.** Let $x \in X$. Then $\varphi(x^{11})$ is a band in $Y$ containing $\varphi(x)$. Thus $\varphi(x^{11}) \supset \varphi(x)^{11}$. Since $\varphi^{-1}$ has the same property, $\varphi^{-1}(\varphi(x)^{11}) \supset (\varphi^{-1}(\varphi(x)))^{11} = x^{11}$. Thus $\varphi(x)^{11} \supset \varphi(x^{11})$ and we obtain $\varphi(x^{11}) = \varphi(x)^{11}$. If $|x| \perp |y| = 0$, then $x^{11} \cap y^{11} = \{0\}$. Hence $\varphi(x)^{11} \cap \varphi(y)^{11} = \{0\}$ and $|\varphi(x)| \perp |\varphi(y)| = 0$.

Now let $C = \{ x \in X^+ : \varphi(x) > 0 \}$. Clearly $C$ is a closed cone in $X$ and $V = C - C$ is a closed linear subspace of $X$. Let $P$ be a band projection on $X$ and $x \in C$. Then $|\varphi(Px)| \perp |\varphi(x - Px)| = 0$ and $0 < \varphi(x) = \varphi(Px) + \varphi(x - Px)$. Thus $\varphi(Px) > 0 \Rightarrow Px \in C$. Hence $P(V) \subset V$ and by Lemma 1, $V$ is a band in $X$. Now suppose $x \in V^\perp$ and $x > 0$. Then $\varphi(x) \leq 0$. For, suppose $\varphi(x) = y_1 - y_2$ with $y_1 > 0$, $y_2 > 0$, $y_1 \perp y_2 = 0$. Then $x = \varphi^{-1}(y_1) - \varphi^{-1}(y_2)$ implies $\varphi^{-1}(y_1) > 0$ and $\varphi^{-1}(y_2) \in V$. Thus we define $\psi$ by $\psi|V = \varphi$ and $\psi|V^\perp = -\varphi$. Then $\psi$ is a linear isometry of $X$ onto $X$ order preserving.

The main basis of our analysis is the theorem of Lamperti [5] on the representation of linear isometries on $L_p(\mu)$. Since we are assuming that $L_p(\mu)$ is separable, we can take $L_p(\mu) = L_p(T, \Sigma, \mu)$ where $T = T_1 \cup T_2$ where either $T_1 = \emptyset$ or $T_1 = [0, 1]$, $T_2 = \emptyset$ or $T_2$ is a countable set in $[2,3]$ (for example). Moreover, $\Sigma|T_1$ can be taken to be the Lebesgue measurable
sets on $T_1$ (if $T_1 \neq \emptyset$) and $\Sigma|T_2$ to be the set of all subsets of $T_2$ (if $T_2 \neq \emptyset$), and $\mu$ can be taken to be Lebesgue measure on $\Sigma|T_1$ (if $T_1 \neq \emptyset$), counting measure on $\Sigma|T_2$ if $T_2$ is finite and nonempty, and if $T_2 = \{t_n\}$, we take $\mu(t_n) = 1/2^n$ (recall the representation theory for separable $L_p(\mu)$ spaces).

**Theorem (Lamperti).** Let $1 < p < \infty$ and $p \neq 2$ and $V$ be a linear isometry of $L_p(\mu)$ onto $L_p(\mu)$. Then there is a measurable function $h$ on $T$ and a measurable $\varphi: T \to T$ such that $\varphi(T_1) = T_1$ (a.e.), $\varphi(T_2) = T_2$, $\varphi$ is essentially one-to-one (i.e., one-to-one except on a set of measure zero), $\varphi^{-1}$ is measurable, and $Vf = h(f \circ \varphi)$ for all $f \in L_p(\mu)$ and $\int \varphi^{-1}(E) |h|^p d\mu = \mu(E)$ for all measurable sets $E \subset T$, and $|h| = 1$ on $T_2$.

If $V^2 = I$ (the identity), then it is easy to see that $\varphi^2 = \text{identity}$ and that $h = 1/h \circ \varphi$ since $1 = V^2(1)$.

Moreover, we can decompose $T$ into three disjoint measurable sets $A_0, A_1, A_2$ where $A_0 = \{t: \varphi(t) = t\}$, $\varphi$ maps $A_1$ onto $A_2$ and $A_2$ onto $A_1$, and $|h| = 1$ on $A_0$, e.g., put $A_1 = \{t: \varphi(t) > t\}$ and $A_2 = \{t: \varphi(t) < t\}$.

The crucial part of our proof depends on an analysis via the Lamperti theorem of the decomposition of $T$ whenever we have two projections $P_1 = \frac{1}{2}(I + V_1)$ and $P_2 = \frac{1}{2}(I + V_2)$ where $V_1$ and $V_2$ are idempotent linear isometries on $L_p(\mu)$ and $P_1$ dominates $P_2$, i.e., $P_1 P_2 = P_2 P_1 = P_2$. From the Lamperti theorem we obtain functions $h_1, h_2, \varphi_1, \varphi_2$ with the properties listed in the theorem and the following remark. Let $A_0, A_1, A_2$ be the above described sets relative to $\varphi_1$. Then on $A_0$ we have that $h_1^2 = 1$. Thus $A_0^+ = A_0 \cup A_0^-$ where $A_0^+ = \{t \in A_0: h_1(t) = 1\}$ and $A_0^- = \{t: h_1(t) = -1\}$ and $\varphi_1$ maps $A_1$ onto $A_2$ and $A_2$ onto $A_1$.

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**Lemma 4.** (A) Let $C = \{t \in A_1 \cup A_2: \varphi_2(t) = t\}$. Then for

$$D = (A_1 \cup A_2) \setminus C$$

we have the following relations (a.e.).

(i) $h_2 = -1$ on $C$,
(ii) $\varphi_1 = \varphi_2$ on $D$,
(iii) $h_1 = h_2$ on $D \cup A_0^-$,
(iv) $\varphi_2$ is the identity on $A_0^-$. 
(B) Let $B_0 = \{ t \in A_0^+ : \varphi_2(t) = t \}$, $B_1 = \{ t \in A_0^+ : \varphi_2(t) > t \}$ and $B_2 = \{ t \in A_0^+ : \varphi_2(t) < t \}$. Then $A_0^+ = B_0 \cup B_1 \cup B_2$, $\varphi_2(B_1) = B_2$, $\varphi_2(B_2) = B_1$.

**Proof.** Since $\varphi_1$ maps $A_1$ onto $A_2$ and $A_2$ onto $A_1$, and $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, it follows that $\varphi_1(C \cap A_1) = C \cap A_2$. Let $f = \varphi_1^{-1}$, the characteristic function of $C \cap A_1$. Then $P_2f = \frac{1}{2}(f + h_2f)$ ($f \circ \varphi_2 = f$) has its support contained in $C \cap A_1$ and

$$P_1P_2f = \frac{1}{4} \left[ f + h_2f + h_1(f \circ \varphi_1 + h_2 \circ \varphi_1 \cdot f \circ \varphi_1) \right] = P_2f.$$  

Thus it follows that $P_2f = 0$ since $h_1(f \circ \varphi_1 + h_2 \circ \varphi_1 \cdot f \circ \varphi_1) = 0$ on $C \cap A_1$. A similar argument shows that $P_2g = 0$ if $g = \varphi_1^{-1}$ on $A_2$. Thus $P_2(f + g) = \frac{1}{2}(f + g + h_2(f + g)) = 0$ and it follows that $h_2 = -1$ on $C$. Hence part (i) of (A) is true.

Suppose that $\varphi_1(D) \neq \varphi_2(D)$. Then there is a subset $Z$ of $D$ with positive measure such that $Z, \varphi_1(Z), \varphi_2(Z)$ are pairwise disjoint (consider $Z = \{ t \in A_1 \cap D : \varphi_1(t) > \varphi_2(t) \}$ or the corresponding sets with $A_1$ replaced by $A_2$ and/or greater than by less than). Let $f = \varphi_1^{-1}$ on $A_1$ and $\varphi_2(Z)$ but $\varphi_1^{-1}(Z)$ so that $P_2f \neq P_1P_2f$. Thus part (ii) of (A) is established.

Since $\varphi_1 = \varphi_2$ on $D$, $\varphi_1^{-1} = \varphi_1$, and $\varphi_2^{-1} = \varphi_2$, from the Lamperti theorem it follows that $|h_1| = |h_2|$ on $\varphi_1(D) = \varphi_2(D)$ and since $h_1h_1 \circ \varphi_1 = h_2h_2 \circ \varphi_2 = 1$, it follows that $|h_1| = |h_2|$ on $D$. Now suppose that $h_1 = -h_2$ on a subset $Z$ of $D$ with positive measure and put $f = \varphi_1^{-1}$ on $A_1$. Then $P_2f = \frac{1}{2}(f + h_2\varphi_1^{-1}(Z \cap A_1))$,

$$P_1P_2f = \frac{1}{4} \left[ f + h_2\varphi_1^{-1}(Z \cap A_1) + h_1f \circ \varphi_1 + h_1(h_2 \circ \varphi_2)\varphi_1^{-1}(Z \cap A_1) \right]$$

$$= \frac{1}{4} \left[ f + h_2\varphi_1^{-1}(Z \cap A_1) + h_1\varphi_1^{-1}(Z \cap A_1) + h_1h_2 \circ \varphi_2f \right]$$

$$= \frac{1}{4} \left[ f + h_2\varphi_1^{-1}(Z \cap A_1) - h_2\varphi_1^{-1}(Z \cap A_1) - f \right] = 0$$

since $h_1 = -h_2$ on $\varphi_1^{-1}(Z \cap A_1)$. A similar result holds for $f = \varphi_2^{-1}$ on $A_2$. Thus we obtain a contradiction to the assumption that $P_1P_2 = P_2$ since one of $Z \cap A_1$ and $Z \cap A_2$ has positive measure. Hence $h_1 = h_2$ on $D$.

If $f \in L_\varphi(\mu)$ and supp $f \subset A_0^+$, then $P_1f = \frac{1}{2}(f + h_1f \circ \varphi_1) = \frac{1}{2}(f - f) = 0$. Thus $P_2f = P_2P_1f = 0$. Since $P_2f = \frac{1}{2}(f + h_2f \circ \varphi_2)$, it follows by putting $f = \varphi_2^{-1}$ that $h_2 = -1$ and $\varphi_2$ is the identity on $A_0^+$. Thus part (iv) of (A) is valid.

Part (B) follows from the observation that $\varphi_2(A_0) = A_0$ since $\varphi_1\varphi_2 = \varphi_2\varphi_1$ and that $\varphi_2$ is the identity on $A_0^+$ so that $\varphi_2(A_0^+) = A_0^+$.

We shall continue the above ordering on projections in the next lemma. Specifically, if we have a sequence of band projections $P_n$ on a Banach lattice $X$ with order continuous norm and if $P_nP_{n+1} = P_{n+1}P_n = P_{n+1}$ for all $n$, then inf $P_n = P$ is also a band projection, where $P(x) = \lim_n P_n(x)$ for all $x \in X$.

Note that since the norm is order continuous, if $x > 0$, $\{P_n(x)\}$ is a decreasing sequence of positive elements and, hence, must converge in norm to its infimum. Thus $P$ can be so defined. It is easy to check that $P$ is a projection onto $\cap_{n=1}^\infty P_n(X) = M$ and that $M$ is a band in $X$. To see that $P$ is the band projection note that for any $x > 0$, $P_x \cap (x - P_x) = \lim_n P_nx \cap (x - P_nx)$.  


We assume that the Banach space $L_p(\mu)$ has a lattice structure which makes it a Banach lattice in the $L_p$-norm for the next lemma.

**Lemma 5.** There is a band projection $P$ which is minimal with respect to property:

\((\ast)\) For each set $E \subset T$ of positive measure there is an $f \in L_p(\mu)$ such that $(Pf)|E \neq 0$.

**Proof.** Since $L_p(\mu)$ is separable it suffices to show that if $P_1 \geq P_2 \geq \cdots$ is a sequence of band projections in the assumed Banach lattice structure satisfying property $(\ast)$, then so does $P = \inf_n P_n$ (the norm is order continuous on the Banach lattice structure under consideration since as a Banach space $L_p(\mu)$ does not contain a linearly isomorphic copy of $c_0$; see [8]).

Let $P_n = \frac{1}{2}(I + V_n)$ where $V_n$ is an idempotent linear isometry on $L_p(\mu)$ and $V_n f = h_n(f \circ \varphi_n)$ where $h_n, \varphi_n$ are given by Lamperti’s theorem. The condition $(\ast)$ is equivalent to $h_n \neq 1$ (a.e.) on sets where $\varphi_n$ is the identity since $(Pf) \cdot \mathcal{X}_E = \frac{1}{2}(f \mathcal{X}_E + h \cdot (f \circ \varphi) \cdot \mathcal{X}_E)$. For, if $h = -1$ on $E$ and $\varphi$ is the identity on $E$, then $(Pf) \cdot \mathcal{X}_E = 0$ for all $f$ and if $(Pf) \cdot \mathcal{X}_E = 0$ for all $f$, then for $f = \mathcal{X}_F$, $\mathcal{X}_F = -h\mathcal{X}_E(f)$ for $F \subset E$, which implies that $h = -1$ on $E$ and $\varphi$ is the identity on $E$.

Let $A_0, n, A_{1,n}, A_{2,n}$ be the sets associated with $P_n$ in the remarks preceding Lemma 4. Then using Lemma 4 we can define $h$ and $\varphi$ by $h(t) = h_n(t)$ and $\varphi(t) = \varphi_n(t)$ if $t \in \limsup_1 A_{1,k} \cup A_{2,k}$ and $n$ is the least integer such that $t \in A_{1,k} \cup A_{2,k}$ and $h(t) = 1$ and $\varphi(t) = t$ if $t \in \liminf_1 A_{0,k}$.

Since $\|h_n\| = 1, \|h\| = 1$ and $Vf = h(f \circ \varphi) = \lim_n V_n f$ is an isometry. Moreover, $P = \frac{1}{2}(I + V)$, and since $h \neq -1$ where $\varphi$ = identity, $P$ satisfies property $(\ast)$.

We now come to the main theorem of the paper.

**Theorem.** Let $X$ be a Banach lattice linearly isometric to $L_p(\mu)$. Then there are measurable sets $A_0, A_1$ in $T$ such that $X$ is linearly order isometric to $L_p(A_0) \oplus_p L_p(A_1, \mathcal{E}_p, 2)$.

**Proof.** Let $P$ be any band projection in $X$ and $A_0, A_1, A_2$ the sets associated with $P$ (see Lemma 4 and the remarks preceding it). Then $L_p(A_1 \cup A_2)$ and $L_p(A_0)$ are bands in $X$ and in each case the band projections are given by multiplication by the associated characteristic functions. We prove this for $L_p(A_1 \cup A_2)$ and the other proof is similar. Suppose $Q$ is a band projection and $Q \leq P$. Then by Lemma 4 it follows that the support of $Q(f \cdot \mathcal{X}_{A_1 \cup A_2})$ is contained in $A_1 \cup A_2$ for all $f \in L_p(\mu)$. If $Q \leq I - P$, then Lemma 4 shows the same result. Thus by Lemma 1, $L_p(A_1 \cup A_2)$ is a band in $X$. Let $Q$ be the band projection of $X$ onto $L_p(\mu)$. Let $h_2, \varphi_2$ be the functions such that $Qf = \frac{1}{2}(f + h_2 f \circ \varphi_2)$ for all $f \in L_p(\mu)$. Then for any measurable set $Z \subset A_1 \cup A_2$,

$$\mathcal{X}_Z = Q(\mathcal{X}_Z) = \frac{1}{2}(\mathcal{X}_Z + h_2 \cdot \mathcal{X}_{\varphi_2(Z)}).$$

Thus $h_2 = 1$ on $Z$ and $\varphi_2(Z) = Z$. By Lemma 4 we conclude that $\varphi_2$ is the identity on $A_1 \cup A_2$. Now for $f \in L_p(\mu),$

$$Qf = \frac{1}{2}(f + h_2 f \circ \varphi_2) \cdot \mathcal{X}_{A_1 \cup A_2} = f \cdot \mathcal{X}_{A_1 \cup A_2}.$$
We now assume that $P$ is minimal with respect to property $(\ast)$. Let $Q \leq P$ and let $B_0, B_1, B_2$ be the sets given by Lemma 4 for $Q$. Then $\mu(B_1 \cup B_2) = 0$. For, if not, then by the above $L_p(B_1 \cup B_2)$ is a band in the range of $P$ and $Qf = Pf - f \cdot \mathcal{N}_{B_1 \cup B_2}$ defines a band projection with property $(\ast)$ which is smaller than $P$. We define $u: L_p(\mu) \to L_p(A_0) \oplus_p L_p(A_1, E_p(2))$ as follows:

$$u(f) = f \cdot \mathcal{N}_{A_0} + [(Pf) \cdot \mathcal{N}_{A_1}(e_1 + e_2) + (f - Pf) \cdot \mathcal{N}_{A_1}(e_1 - e_2)].$$

A routine calculation shows that $u$ is a linear isometry and onto. By Lemma 3 we only need to show that $u$ and $u^{-1}$ both carry bands onto bands. If $Q$ is a band projection and $Q \leq P$, then let $B_0, B_1, B_2$ be the sets in Lemma 4 for $Q$. Then for $f \in L_p(\mu)$,

$$(Qf) \cdot \mathcal{N}_{A_0} = f \cdot \mathcal{N}_{B_0^*}$$

and

$$(Qf) \cdot \mathcal{N}_{A_1} = \frac{1}{2} (f \cdot \mathcal{N}_{A_1 \cap D} + (h_1 f \circ \varphi_1) \cdot \mathcal{N}_{A_1 \cap D})$$

where $h_1, \varphi_1$ are the functions associated with $P$. That is,

$$Q(X) = L_p(B_0^+) \oplus_p L_p(A_1 \cap D) (e_1 + e_2).$$

If $Q \leq I - P$, then $I - Q \leq P$ and if $B_0$ and $D$ are the sets for $I - Q$ as above, then a similar argument shows that

$$Q(X) = L_p(B_0^+) \oplus_p L_p(A_1 \cap D) (e_1 + e_2).$$

If $Q \leq I - P$, then $I - Q \leq P$ and if $B_0$ and $D$ are the sets for $I - Q$ as above, then a similar argument shows that

$$Q(X) = L_p(B_0^-) \oplus_p L_p(D) (e_1 + e_2).$$

If $(g_1, g_2, g_3) \in L_p(A_0) \oplus_p L_p(A_1, E_p(2))$, then $f = g \cdot \mathcal{N}_{A_0} + (g_2 + g_3) \cdot \mathcal{N}_{A_1} + [h_1(g_2 - g_3) \circ \varphi_1] \cdot \mathcal{N}_{A_2}$. $u(f) = g$ and proofs similar to the above show that $u^{-1}$ carries bands onto bands.

**ADDED IN PROOF.** The first author and S. Bernau have recently shown that the same theorem holds in the general case. This was done by characterizing bicontractive projections on $L_p(\mu)$ as those projections such that $2P - I$ is an isometry.

**REFERENCES**

4. E. Lacey and P. Wojtaszczyk, *Non-atomic M spaces can have $l_1$ as a dual space* (to appear).