ISOMETRY GROUPS OF MANIFOLDS OF NEGATIVE CURVATURE

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Abstract. Solvable subgroups of the isometry groups of a simply-connected manifold of negative curvature are characterized and this characterization is used to show that the isometry group of the universal Riemannian covering of a compact manifold of negative curvature is either discrete or semisimple.

0. Introduction. A number of recent papers have related the geometry of manifolds of negative curvature to the algebra of various groups of isometries (for example [4], [8]). In this paper we study various groups of isometries of a simply-connected manifold $M$ of negative curvature. In Theorem 5 we use results of Bishop and O'Neill [2] to show that a solvable group of isometries either leave a single geodesic invariant, permute a class of asymptotic geodesics, or else have a nonempty fixed point set. If the total isometry group $I(M)$ does not satisfy either of two former conditions, we show in Theorem 7 that there is a compact normal subgroup $K$ such that $I(M)/K$ is semisimple and acts effectively on a closed, connected, totally convex submanifold of $M$. Using these results we show in Theorem 9 that if $M$ is the universal Riemannian covering of a compact manifold of negative curvature, then the isometry group $I(M)$ is either discrete or semisimple. This may be viewed as an extension of the classical situation where the compact manifold may be considered as a double coset space $\Gamma \backslash G / K$ of a connected semisimple Lie group $G$ and where the symmetric space $G / K$ can be given an invariant metric of nonpositive curvature so that $G$ is isomorphic to the identity component of the isometry group [5].

1. Preliminaries. $M$ will denote a simply-connected, complete, Riemannian manifold of sectional curvature $K \leq C < 0$. Given any oriented geodesic $\gamma$ and any point $X \in M$ there exists a unique oriented geodesic through $x$ whose distance from $\gamma$ tends to zero as $t$ tends to $\infty$, the asymptote to $\gamma$ through $x$ [2]. Orthogonal trajectories to a family of asymptotic geodesics give a foliation of $M$ by $(n - 1)$ planes called horospheres [1].

$I(M)$ will denote the Lie group of isometries of $M$ and $I_0$ its identity component. If $\xi$ is a class of asymptotic geodesics let $S(\xi)$, the stability group of $\xi$, be the subgroup of isometries which permute the geodesics of $\xi$ (compare

Received by the editors September 24, 1973 and, in revised form, October 29, 1974.


Key words and phrases. Riemannian manifold, negative curvature, isometry group.

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'ray-subgroup' [2]). Now each element of $S(\xi)$ will also permute the horospheres associated to $\xi$. Define $H(\xi)$ to be the subgroup of $S(\xi)$, composed of isometries which map each such horosphere into itself. However using the fact that isometries of $S(\xi)$ commute with the geodesic flow restricted to geodesics of $\xi$ we can show

**Lemma 1.** If $\phi \in S(\xi)$ maps one horosphere associated to $\xi$ into itself, then $\phi \in H(\xi)$.

We need the following characterization of isometries on $M$.

**Proposition 2 (Bishop-O'Neill [2]).** Let $\phi$ be an isometry of $M$. Then exactly one of the following is true:

(a) $\phi$ has a fixed point.
(b) $\phi$ translates a (unique) geodesic.
(c) $f_\phi : M \to R$ defined by $f_\phi(x) = d^2(x,\phi(x))$ has no minimum.

Following [3], we call the isometries (a), (b), and (c) elliptic, axial, and parabolic, respectively. Parabolic isometries preserve a unique class of asymptotic geodesics, whereas axial isometries preserve exactly two such classes. We show

**Lemma 3.** If a parabolic isometry $\phi$ is in the stability group $S(\xi)$, it must also be in $H(\xi)$.

**Proof.** We show that if $\phi \notin H(\xi)$, then it must be axial. If $\varphi_t : M \to M$ is the geodesic flow along geodesics of $\xi$ and $H(x)$ is the horosphere passing through $x \in M$, then $\phi(H^+(x)) = \phi_{t_0}(H^+(x))$ for some $t_0 \in R$. Now the fact that $M$ has curvature bounded above by a constant less than zero implies (Arnold-Avez [1]) that $\varphi_{t_0} : H^+(x) \to H^+(\varphi_{t_0}x)$ is a metric space contraction mapping (if $\varphi_{t_0}$ is an expansion we consider $\varphi_{-t_0}$). Since $\phi$ is an isometry, $\varphi_{-t_0} \circ \varphi_{t_0} : H^+(x) \to H^+(x)$ is also a contraction and has a unique fixed point by the contraction mapping theorem for complete metric spaces. This fixed point corresponds to the geodesic translated by $\phi$.

2. **Solvable and nilpotent subgroups of $I(M)$**. If $S$ is any subset of $I(M)$ we denote by Fix $S$ the set of all common fixed points of elements of $S$. Then Fix $S$ is known to be a closed totally geodesic submanifold of $M$.

A set $K \subseteq M$ is totally convex if whenever $x$ and $y$ are two points of $K$ the infinite geodesic joining $x$ to $y$ lies in $K$. A totally convex submanifold is necessarily connected and totally geodesic.

**Lemma 4.** Let $S$ be a solvable group of elliptic isometries of $M$. Then Fix $S$ is nonempty.

**Proof.** We first prove the lemma for $S$ abelian. Consider closed, totally convex submanifolds of $M$ which are invariant under the isometries of $S$. Let $\phi$ be any isometry in $S$ and let $C_\phi = \text{Fix} \phi$, which is nonempty by assumption. Then $C_\phi$ is a closed, totally convex submanifold which is invariant under $S$ because $S$ is abelian. Now let $C$ be a submanifold with the above properties which is of minimal dimension. Since $C_\phi$ is invariant under $\phi$, $C$ must intersect $C_\phi$ nontrivially [4, Lemma 1]. However, since $C \cap C_\phi$ is a submanifold of $C$,
the minimality of $C$ implies that $C$ is contained in $C_\phi$ for all $\phi \in S$ and, in particular, that $Fix \ S$ is nonempty.

In general let $S = S_0 \supset S_1 \supset \cdots \supset S_{k-1} \supset \{1\}$ be the derived series for $S$. Fix $S_{k-1}$ is nonempty since $S_{k-1}$ is abelian. Now $S_{k-2}$ leaves $Fix \ S_{k-1}$ invariant because $S_{k-1}$ is normal in $S_{k-2}$. Thus $S_{k-2}/S_{k-1}$ is an abelian group of isometries of $Fix \ S_{k-1}$ and the above reasoning implies that $Fix \ S_{k-2}$ is nonempty. Continuing in this way we show that $Fix \ S$ is nonempty.

**Theorem 5.** Let $S$ be a solvable group of isometries with no common fixed points. Then either $S$ leaves some geodesics invariant or else $S$ is contained in $S(\xi)$ for some class $\xi$ of asymptotic geodesics.

**Proof.** Let $S = S_0 \supset S_1 \supset \cdots \supset S_{n-1} \supset \{1\}$ be the derived series for $S$ and let $S_k$ be the largest group in the sequence consisting entirely of elliptic isometries. Then $Fix \ S_k$ is nonempty and invariant under $S_{k-1}$ which must contain either an axial or a parabolic isometry.

Suppose $\phi \in S_{k-1}$ has axis $\gamma$. Then $\gamma$ lies in $Fix \ S_k$ [4]. Let $\psi$ be any element of $S_{k-1}$. Then $\psi^{-1}\phi\psi^{-1} = \chi \in S_k$. Thus $\psi^{-1}\phi\psi$ leaves $\gamma$ invariant and $\phi$ leaves $\phi(\gamma)$ invariant. However $\phi$ has a unique axis because the curvature of $M$ is strictly negative and so $\psi$ also leaves $\gamma$ invariant. Similarly if $\psi \in S_{k-2}$, consideration of $\psi^{-1}\phi\psi^{-1}$ as above will show that $\psi$ leaves $\gamma$ invariant. Proceeding in this way $S$ must leave $\gamma$ invariant in this case.

Suppose now that $\phi \in S_{k-1}$ is parabolic and that $\xi$ is the unique class of asymptotes permuted by $\phi$. Reasoning exactly as above with $\xi$ taking the place of $\gamma$, one can show that $S$ leaves $\xi$ invariant, i.e. that $S$ is contained in $S(\xi)$.

In the case of a nilpotent group we can obtain results which are a little more precise.

**Theorem 6.** Let $N$ be a nilpotent group of isometries with no common fixed points. Then either $N$ contains an axial isometry and leaves its axis invariant or else $N$ is contained in $H(\xi)$ for some class $\xi$.

**Proof.** Consider the series $N = N_0 \supset N_1 \supset \cdots \supset N_{n-1} \supset \{1\}$ where $N_t = [N, N_{t-1}]$ and again suppose that $N_k$ is the largest group for which $Fix \ N_t$ is nonempty. As above, the existence of an axial isometry $\phi$ in $N_{k-1}$ implies that $N$ leaves the axis of $\phi$ invariant.

Now suppose that $\phi \in N_{k-1}$ is parabolic with $\phi \in H(\xi)$. As in Theorem 5 we can show that $N$ is contained in $S(\xi)$. Suppose that there exists $\psi \in N$ such that $\psi \not\in H(\xi)$. Then (Lemma 3) $\psi$ must be axial with axis $\gamma$, say. Now $\phi^{-1}\psi\phi^{-1} = \chi \in N_k$ and so $\phi$ must leave $\gamma$ invariant. This is impossible since $\phi$ is parabolic and thus $N$ is contained in $H(\xi)$.

**Note.** Theorem 6 is false for solvable groups. For example when $SL(2, r)$ acts as isometries of the Poincaré upper half plane, the upper triangular matrices form a solvable subgroup containing both axial and parabolic isometries.

3. Invariant manifolds and semisimple groups of isometries. In this section we shall restrict our attention to metrics of $M$ which have the property that the total isometry group $I(M)$ does not leave invariant any one geodesic and is not equal to any single stability group $S(\xi)$. This condition is satisfied, for example,
when $M$ is the Riemannian covering of a negatively curved manifold $N$ with two distinct closed geodesics and, in particular, when $N$ is compact.

**Theorem 7.** Suppose $M$ has a metric which satisfies the above condition on $I(M)$. Then there exists a compact normal subgroup $K$ of $I(M)$ such that either $I(M)/K$ is discrete or else it is semisimple and acts effectively on a closed, totally convex submanifold of $M$.

**Proof.** Let $M'$ be a closed, totally convex submanifold of $M$ which is invariant under all isometries and is of minimal dimension. Let $K$ be the subgroup of $I(M)$ made up of all isometries leaving $M'$ pointwise fixed. $K$ is compact since the subgroup of $I(M)$ leaving any given point fixed is compact [5]. Now suppose $\phi \in I(M)$ and $\psi \in K$ and $x \in M'$ are arbitrary. Then $\phi(x) \in M'$ and so $\psi(\phi(x)) = \phi(x)$. Thus $\phi^{-1}\psi(x) = x$ and so $\phi^{-1}\psi \in K$. Thus $K$ is normal in $I(M)$.

Suppose $I(M)/K$ is not discrete and let $R$ be any normal solvable subgroup. Then $R$ acts as a group of isometries of $M'$. According to Theorem 5 if $\text{Fix } R$ is empty, either $R$ leaves some geodesic of $M'$ invariant or else $R \subseteq S(\xi)$ for some $\xi$. In the former case $R$ contains an axial isometry $\phi$ with axis $g$. The normality of $R$ in $I/K$ implies that $\phi$ preserves the geodesic $a(g)$ for a given $a \in I$ and thus $a(g) = g$ as in the proof of Theorem 5. Thus $I(M)$ preserves $g$. In the latter case it follows similarly that $I(M) = S(\xi)$. Since $\text{Fix } R$ is also closed, and totally convex, the minimality of $M'$ implies that $\text{Fix } R = M'$ and so $R$ is trivial. Thus $I(M)/K$ is semisimple.

**Corollary 8.** If $I(M)$ satisfies the hypothesis of Theorem 7 and $M = M'$ (i.e. except for $M$ there are no closed, totally convex submanifolds invariant under $I(M)$), then $I(M)$ is either semisimple or discrete.

**Theorem 9.** Let $N$ be a compact Riemannian manifold of negative curvature and $M$ its universal Riemannian covering. Then $I(M)$ is discrete or semisimple.

**Proof.** We consider the fundamental group $\Pi_1(N)$ as a subgroup of $I(M)$. Since $N$ is compact, each covering transformation is axial. If $\Pi_1(N)$ were to leave a single geodesic invariant or if $\Pi_1(N)$ were contained in $S(\xi)$ for some $\xi$, it would then follow [2] that $\Pi_1(N)$ is isomorphic to the integers, which is impossible for compact $N$. [6]. Thus $I(M)$ must satisfy the hypothesis of Theorem 7. Now suppose there exists a closed, connected, totally convex submanifold $M'$—not $M$ itself—which is invariant under $I(M)$ and thus under $\Pi_1(N)$. Let $N' = p(M')$ where $p : M \to N$ is the covering projection. Then $N'$ is closed and totally convex and so $N$ is diffeomorphic to the normal bundle of $N'$ [2, Lemma 3.1] and, in particular, is noncompact. Thus $M' = M$ and the result follows from Corollary 8.

**References**


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