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EXTENSIONS OF CONTINUOUS FUNCTIONS FROM DENSE SUBSPACES

ROBERT L. BLAIR

ABSTRACT. Let X and Y be topological spaces, let S be a dense subspace of X, and let $f: S \to Y$ be continuous. When Y is the real line $\mathbf{R}$, the Lebesgue sets of f are used to provide necessary and sufficient conditions in order that the (bounded) function f have a continuous extension over X. These conditions yield the theorem of Taĭmanov (resp. of Engelking and of Blefko and Mrówka) which characterizes extendibility of f for Y compact (resp. realcompact). In addition, an extension theorem of Blefko and Mrówka is sharpened for the case in which X is first countable and Y is a closed subspace of $\mathbf{R}$.

We first quote (in Theorem 1) two basic results concerning extension of a continuous function from a dense subspace of a topological space. Theorem 1A is due to Taĭmanov [10] (see also [5, Theorem 3.2.1]) and, in dual form, to Eilenberg and Steenrod [3, Lemma 10.9.6] (cf. [5, Exercise 3.2A]). Theorem 1B is due, independently, to Engelking [4, Theorem 2] and to Blefko and Mrówka [2, Theorem A]. (Theorem A of [2] includes the unneeded hypothesis that X is T_1.) For additional results on extension of continuous functions from dense subspaces, see McDowell [7].

THEOREM 1. *Let X and Y be topological spaces, let S be a dense subspace of X, and let $f: S \to Y$ be continuous.*

A (TAĬMANOV). *If Y is compact (Hausdorff), then these are equivalent*:

(1) *f extends continuously over X.*

(2) *If F_1 and F_2 are disjoint closed subsets (or, alternatively, zero-sets) of Y, then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ have disjoint closures in X.*

B (ENGELKING AND BLEFKO-MRÓWKA). *If Y is realcompact, then these are equivalent*:

(1) *f extends continuously over X.*

(2) *If $\{F_n\}_{n=1}^{\infty}$ is any sequence of closed subsets (or, alternatively, zero-sets) of Y with $\bigcap_{n=1}^{\infty} F_n = \varnothing$, then $\bigcap_{n=1}^{\infty} \mathrm{cl}_X f^{-1}(F_n) = \varnothing$.*

By a *zero-set* is meant the set of zeros of a real-valued continuous function. For the theory of realcompact spaces, see Gillman and Jerison [6].

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In this note we obtain a sharper version of Theorem 1A (resp. 1B) for the special case in which f is a bounded continuous function (resp. continuous function) from S into the real line $\mathbf{R}$; this is Theorem 2 below. Theorem 1, in turn, will follow readily from Theorem 2. We also include a sharpening (for real-valued functions) of a theorem of Blefko and Mrówka concerning extension of a continuous function from a dense subspace of a first countable space [2, Theorem D] (see Theorem 3 below).

If X is a topological space, then $C(X)$ (resp. $C^*(X)$) will denote the set of all continuous (resp. bounded continuous) real-valued functions on X. If $f \in C(X)$ and $a \in \mathbf{R}$, we set

$$L_a(f) = \{x \in X : f(x) \leqslant a\}, \qquad L^a(f) = \{x \in X : f(x) \geqslant a\}.$$

Sets of the form $L_a(f)$ or $L^a(f)$ are *Lebesgue sets* of f. The point of Theorem 2 (which may be viewed as an analogue of [6, 1.18]) is that it characterizes extendibility of f in terms of the Lebesgue sets of f. (Theorem 2 is thus a fragment of a general program whereby real-valued functions are studied by means of their Lebesgue sets; see, e.g., [8], [9], and [1, §§2–3]. Other aspects of this program will be treated by the author elsewhere.)

THEOREM 2. *Let S be a dense subspace of a topological space X, let $f \in C(S)$, and consider these conditions on f:*

(a) *f extends continuously over X.*

(b) *Disjoint Lebesgue sets of f have disjoint closures in X.*

(c) $\bigcap_{n=1}^{\infty} \mathrm{cl}_X(L_{-n}(f) \cup L^n(f)) = \varnothing$.

Then (a) is equivalent to the conjunction of (b) *and* (c); *and if $f \in C^*(S)$,* (a) *is equivalent to* (b).

PROOF. First assume (a), so that $f = g|S$ for some $g \in C(X)$. To verify (b), we need only note that if $a < b$, then

$$\mathrm{cl}_X L_a(f) \cap \mathrm{cl}_X L^b(f) \subset L_a(g) \cap L^b(g) = \varnothing.$$

To verify (c), let $p \in X$, choose $n \geqslant |g(p)| + 1$, and note that $\{x \in X: |g(x) - g(p)| < 1\}$ is a neighborhood of p in X which misses $L_{-n}(f) \cup L^n(f)$.

Observe next that, to verify (a), it suffices to show that f has an extension $f_p \in C(S \cup \{p\})$ for every $p \in X$. (For then $g : X \to \mathbf{R}$ can be defined by $g = f$ on S and $g(p) = f_p(p)$ for $p \in X - S$; and since S is dense in X, g is continuous [6, 6H].) For the remainder of the proof, we may therefore assume that $X = S \cup \{p\}$, with $p \notin S$.

Assume (b) and (c), and let $A = \{s \in \mathbf{R} : p \in \mathrm{cl}\, L^s(f)\}$, $B = \{r \in \mathbf{R} : p \in \mathrm{cl}\, L_r(f)\}$. Since $X = \mathrm{cl}\, S$, (c) implies that there is an n such that $p \in \mathrm{cl}\, L_n(f) \cap \mathrm{cl}\, L^{-n}(f)$. Hence both A and B are nonempty. Moreover, by (b), we have $s \leqslant r$ for every $s \in A$ and every $r \in B$. Let $s^* = \sup A$, $r^* = \inf B$, and note that $s^* \leqslant r^*$. If $s^* < r^*$, there is $t \in \mathbf{R}$ with $s^* < t < r^*$; but then $p \notin \mathrm{cl}(L_t(f) \cup L^t(f)) = \mathrm{cl}\, S$, a contradiction. Thus $s^* = r^*$. Define $g : X \to \mathbf{R}$ by $g = f$ on S and $g(p) = s^* = r^*$. We verify that g is continuous at each point of X:

Let $x \in X$, $\epsilon > 0$, and $V = (g(x) - \epsilon, g(x) + \epsilon)$.

Case 1. $x = p$. Let $U = X - \mathrm{cl}(L_{g(p)-\epsilon}(f) \cup L^{g(p)+\epsilon}(f))$. Since $g(p) - \epsilon < s \leqslant r < g(p) + \epsilon$ for some $s \in A$ and $r \in B$, it follows from (b) that $p \in U$, and clearly $g(U) \subset V$.

Case 2. $x \in S$. There is an open neighborhood W of x in S with $f(W) \subset (g(x) - (\epsilon/3), g(x) + (\epsilon/3))$. Write $W = S \cap G$, with G open in X. If $p \notin G$, then $g(G) = f(W) \subset V$, so we may assume that $p \in G$. Now $W \subset L^{g(x)-(\epsilon/3)}(f)$, so we have $p \in \mathrm{cl}\, G = \mathrm{cl}\, W \subset \mathrm{cl}\, L^{g(x)-(\epsilon/3)}(f)$. If $g(p) < g(x) - (2\epsilon/3)$, there is $r \in B$ with $r < g(x) - (2\epsilon/3)$. But then $p \in \mathrm{cl}\, L_r(f) \subset \mathrm{cl}\, L_{g(x)-(2\epsilon/3)}(f)$, which is contrary to (b). Thus $g(p) \geqslant g(x) - (2\epsilon/3)$, and, similarly, $g(p) \leqslant g(x) + (2\epsilon/3)$. We conclude that $g(G) \subset V$, and hence g is a continuous extension of f.

To complete the proof, note that if $f \in C^*(S)$ and if $n > |f|$, then $L_{-n}(f) \cup L^n(f) = \varnothing$, so (c) holds automatically.

Proof of Theorem 1. A(1) $\Rightarrow$ A(2). Assume that $f = g|S$, with $g : X \to Y$ continuous. If F_1 and F_2 are disjoint closed subsets of Y, then

$$\mathrm{cl}_X f^{-1}(F_1) \cap \mathrm{cl}_X f^{-1}(F_2) \subset g^{-1}(F_1) \cap g^{-1}(F_2) = \varnothing.$$

Similarly, B(1) $\Rightarrow$ B(2).

A(2) $\Rightarrow$ A(1) (resp. B(2) $\Rightarrow$ B(1)). We may assume that the compact (resp. realcompact) space Y is a closed subspace of a product $Y' = \prod_{\alpha \in I} Y_\alpha$, where $Y_\alpha = [0, 1]$ (resp. $Y_\alpha = \mathbf{R}$) for each $\alpha \in I$ (see [6, 11.12]). Let $f_\alpha = (\mathrm{pr}_\alpha | Y) \circ f$, where pr_α is the projection of the product Y' of index α. It suffices to show that each f_α satisfies (b) (resp. (b) and (c)) of Theorem 2. (For then f_α has a continuous extension $g_\alpha : X \to Y_\alpha$, the diagonal map $g = \Delta_{\alpha \in I} g_\alpha : X \to Y'$ is continuous, $g = f$ on S, and $g(X) = g(\mathrm{cl}\, S) \subset \mathrm{cl}\, g(S) \subset Y$; cf. [4, Lemma 1].) For each $a \in \mathbf{R}$, let $Z_a = Y \cap \mathrm{pr}_\alpha^{-1}((-\infty, a])$, $Z^a = Y \cap \mathrm{pr}_\alpha^{-1}([a, +\infty))$. Note that Z_a and Z^a are zero-sets in Y and that $L_a(f_\alpha) = f^{-1}(Z_a)$, $L^a(f_\alpha) = f^{-1}(Z^a)$. It follows from (the zero-set formulation of) either A(2) or B(2) that if $a < b$, then $L_a(f_\alpha)$ and $L^b(f_\alpha)$ have disjoint closures in X; hence (b) holds in either case. Moreover, $\bigcap_{n=1}^\infty (Z_{-n} \cup Z^n) = \varnothing$, so (the zero-set formulation of) B(2) implies that $\bigcap_{n=1}^\infty \mathrm{cl}_X(L_{-n}(f_\alpha) \cup L^n(f_\alpha)) = \varnothing$. Thus (c) holds, and the proof is complete.

We note that, by a similar argument, Theorem C of [2] is also an easy consequence of Theorem 2.

A subset S of a topological space X is *C^*-embedded* (resp. *C-embedded*) in X in case every $f \in C^*(S)$ (resp. $f \in C(S)$) has a continuous extension over X. The following corollary (formulated and proved in [6, Theorems 6.4 and 8.6] in the context of Tychonoff spaces; cf. [11]) is an immediate consequence of either Theorem 1 or Theorem 2.

Corollary. *Let S be a dense subspace of a topological space X.*

A. *These are equivalent*:

(1) *S is C^*-embedded in X.*

(2) *Any two disjoint zero-sets in S have disjoint closures in X.*

B. *These are equivalent*:

(1) *S is C-embedded in X.*

(2) *If a countable family of zero-sets in S has empty intersection, then their closures in X have empty intersection.*

It is known that (the closed set formulation of) Theorem 1A holds if Y is merely Tychonoff, provided that X is first countable [2, Theorem D]. For the special case in which Y is a closed subset of $\mathbf{R}$, we can apply Theorem 2 to sharpen this result as follows:

THEOREM 3. *Let S be a dense subspace of a topological space X, assume each $p \in X - S$ has a countable base of neighborhoods, let Y be a closed subspace of $\mathbf{R}$, and let $f : S \to Y$ be continuous. Then these are equivalent*:

(1) *f extends continuously over X.*

(2) *If F_1 and F_2 are disjoint countable closed subsets of Y, then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ have disjoint closures in X.*

PROOF. (1) $\Rightarrow$ (2). This follows as in the proof of A(1) $\Rightarrow$ A(2) of Theorem 1.

(2) $\Rightarrow$ (1). It suffices to show that f (regarded as a function from S into $\mathbf{R}$) has a continuous extension $g : X \to \mathbf{R}$. (For then $g(X) = g(\mathrm{cl}\, S) \subset \mathrm{cl}\, g(S) \subset Y$.) We verify that $f : S \to \mathbf{R}$ satisfies (b) and (c) of Theorem 2.

Suppose first that (b) fails. Then for some $a < b$ there exists $p \in \mathrm{cl}\, L_a(f) \cap \mathrm{cl}\, L^b(f)$. Obviously $p \in X - S$, so p has a countable base of neighborhoods $\{U_n\}_{n=1}^{\infty}$. Choose $c \in \mathbf{R}$ with $a < c < b$. We shall show that there is a countable closed subset F_1 of $\mathbf{R}$ with $p \in \mathrm{cl}\, f^{-1}(F_1 \cap Y)$ and $F_1 \subset (c, +\infty)$. Let $s^* = \sup\{s \in \mathbf{R} : p \in \mathrm{cl}\, L^s(f)\}$.

Case 1. $s^* < +\infty$. For each $n > 0$, we have $c \vee (s^* - (1/n)) < s^*$, so there is $s(n) \in \mathbf{R}$ with $p \in \mathrm{cl}\, L^{s(n)}(f)$ and $c \vee (s^* - (1/n)) < s(n)$. Moreover, $p \notin \mathrm{cl}\, L^{s^*+(1/n)}(f)$, so there exists a point x_n with

$$x_n \in U_n \cap (X - \mathrm{cl}\, L^{s^*+(1/n)}(f)) \cap L^{s(n)}(f).$$

Let $F_1 = \{f(x_n) : n = 1, 2, \ldots\} \cup \{s^*\}$. Since $|f(x_n) - s^*| < 1/n$, we have $f(x_n) \to s^*$, and hence F_1 is closed in $\mathbf{R}$. Clearly $p \in \mathrm{cl}\, f^{-1}(F_1 \cap Y)$ and $F_1 \subset (c, +\infty)$.

Case 2. $s^* = +\infty$. Construct a sequence $\{x_n\}_{n=1}^{\infty}$ as follows: Pick $x_1 \in U_1 \cap L^b(f)$; and if $x_1, \ldots, x_{n-1}$ have already been chosen with $x_i \in U_i \cap L^b(f)$ and $f(x_i) > f(x_{i-1}) \vee i$ $(i = 2, \ldots, n-1)$, choose $s \in \mathbf{R}$ with $p \in \mathrm{cl}\, L^s(f)$ and $f(x_{n-1}) \vee n < s$, and pick $x_n \in U_n \cap L^s(f)$. Then $\{f(x_n)\}_{n=1}^{\infty}$ is strictly increasing and $f(x_n) \to +\infty$, so $F_1 = \{f(x_n) : n = 1, 2, \ldots\}$ is closed in $\mathbf{R}$. Moreover, $x_n \in U_n \cap L^b(f)$ for all n, so we have $p \in \mathrm{cl}\, f^{-1}(F_1 \cap Y)$ and $F_1 \subset (c, +\infty)$.

Similarly, there is a countable closed subset F_2 of $\mathbf{R}$ with

$$p \in \mathrm{cl}\, f^{-1}(F_2 \cap Y) \quad \text{and} \quad F_2 \subset (-\infty, c).$$

Thus (2) fails.

Suppose next that (c) of Theorem 2 fails. Then there exists $p \in \bigcap_{n=1}^{\infty} \mathrm{cl}_X(L_{-n}(f) \cup L^n(f))$, and clearly $p \in X - S$. Let $\{U_n\}_{n=1}^{\infty}$ be a countable base of neighborhoods at p with $U_n \supset U_{n+1}$ for each n. Pick

$x_1 \in U_1 \cap (L_{-1}(f) \cup L^1(f))$; and if $x_1, \ldots, x_{n-1}$ have already been chosen with $x_i \in U_i \cap (L_{-i}(f) \cup L^i(f))$ and $|f(x_i)| > |f(x_{i-1})| \vee i$ $(i = 2, \ldots, n - 1)$, let $m(n)$ be the least integer $> |f(x_{n-1})| \vee n$, and pick $x_n \in U_n \cap (L_{-m(n)}(f) \cup L^{m(n)}(f))$. We thus construct a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in U_n$, $\{|f(x_n)|\}_{n=1}^{\infty}$ strictly increasing, and $|f(x_n)| \to \infty$. Let

$$F_1 = \{r \in \mathbf{R} : |r| = |f(x_n)| \text{ for some } n, n \text{ odd}\},$$

$$F_2 = \{r \in \mathbf{R} : |r| = |f(x_n)| \text{ for some } n, n \text{ even}\}.$$

Then F_1 and F_2 are disjoint countable closed subsets of $\mathbf{R}$ with

$$p \in \operatorname{cl} f^{-1}(F_1 \cap Y) \cap \operatorname{cl} f^{-1}(F_2 \cap Y),$$

so (2) fails once again. The proof is therefore complete.

We leave open the question of possible generalizations of Theorem 3 (for Tychonoff spaces Y that are not necessarily closed subspaces of $\mathbf{R}$).

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Department of Mathematics, Ohio University, Athens, Ohio 45701