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ON THE LEFSCHETZ NUMBER FOR ITERATES OF CONTINUOUS MAPPINGS[1]

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ABSTRACT. We give an elementary proof of the (mod p)-congruence between the Lefschetz numbers of a continuous mapping and its pth iterate, p prime, and deduce some results about periodic mappings.

The following proposition can be obtained in a less general situation from theorems of Zabreĭko and Krasnosel'skiĭ [10] and Steinlein [9] by using the normalization property of the fixed point index. However, the proofs in [9] and [10] are analytical and quite difficult. Moreover, the normalization property requires some deeper considerations (cf. Dold [1]). In this paper we present a short and appropriate proof using only basic algebraic and topological arguments.

PROPOSITION. *Let X be a compact Hausdorff space such that $\check{H}^*(X;\mathbf{Z})$ is finitely generated. Then $\Lambda(f) \equiv \Lambda(f^p) \mod p$, whenever f is a continuous map on X and p is prime.*

For the proof of the proposition we need the following basic

LEMMA. *Let A be a matrix of integers and let p be prime. Then* trace $A \equiv$ trace $A^p \bmod p$.

PROOF. If A is in $\mathbf{Z}^{n^2}$, we denote by $\Pi : \mathbf{Z}^{n^2} \to \mathbf{Z}_p^{n^2}$ and by $\pi : \mathbf{Z} \to \mathbf{Z}_p$ the corresponding projections. We have $\Pi(A^p) = (\Pi(A))^p$ and $\pi(\text{trace } A) = \text{trace } \Pi(A)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\Pi(A)$. Since the λ_i are roots of a polynomial with coefficients in $\mathbf{Z}_p$, then $\sum_{i=1}^n \lambda_i$ is defined in the extension field $\mathbf{Z}_p(\lambda_1, \ldots, \lambda_n)$ of $\mathbf{Z}_p$. Notice that $\mathbf{Z}_p(\lambda_1, \ldots, \lambda_n)$ is a field of characteristic p, hence $(\sum_{i=1}^n \lambda_i)^p = \sum_{i=1}^n \lambda_i^p$. Since trace $\Pi(A) = \sum \lambda_i$ and trace $(\Pi(A))^p = \sum \lambda_i^p$ are in $\mathbf{Z}_p$, we have $\sum \lambda_i = (\sum \lambda_i)^p = \sum \lambda_i^p$ in $\mathbf{Z}_p$. Thus we get

$$\pi(\text{trace } A) = \text{trace } \Pi(A) = \text{trace } (\Pi(A))^p = \pi(\text{trace } A^p). \quad \square$$

PROOF OF PROPOSITION. From Spanier [8, p. 335], we have that the following diagram with exact rows commutes:

$$\begin{array}{ccccccc}
0 & \to & \check{H}^n(X;\mathbf{Z}) \otimes \mathbf{Q} & \xrightarrow{J} & \check{H}^n(X;\mathbf{Q}) & \to & \operatorname{Tor}(\check{H}^{n+1}(X;\mathbf{Z}),\mathbf{Q}) \\
 & & \downarrow {}_{\mathbf{Z}}f_n^* \otimes \mathrm{id}_{\mathbf{Q}} & & \downarrow {}_{\mathbf{Q}}f_n^* & & \downarrow \\
0 & \to & \check{H}^n(X;\mathbf{Z}) \otimes \mathbf{Q} & \xrightarrow{J} & \check{H}^n(X;\mathbf{Q}) & \to & \operatorname{Tor}(\check{H}^{n+1}(X;\mathbf{Z}),\mathbf{Q})
\end{array}$$

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Since $\operatorname{Tor}(A, \mathbf{Q}) = 0$ for any abelian group A, we have that J is an isomorphism and, therefore,

$$\operatorname{trace}({}_{\mathbf{Z}}f_n^* \otimes \mathrm{id}_{\mathbf{Q}}) = \operatorname{trace}(J^{-1} \circ {}_{\mathbf{Q}}f_n^* \circ J) = \operatorname{trace}({}_{\mathbf{Q}}f_n^*).$$

Let F be a free subgroup and T the torsion subgroup of $\check{H}^n(X; \mathbf{Z})$ such that $\check{H}^n(X; \mathbf{Z}) = F \oplus T$, and let $e_1, \ldots, e_r$ be a basis for F. Then a basis for $\check{H}^n(X; \mathbf{Z}) \otimes \mathbf{Q}$ is given by $e_1 \otimes 1, \ldots, e_r \otimes 1$. If we choose a_{ij} in $\mathbf{Z}$ and t in T such that ${}_{\mathbf{Z}}f_n^*(e_i) = \sum_{j=1}^r a_{ij} e_j + t$, then we have

$${}_{\mathbf{Z}}f_n^* \otimes \mathrm{id}_{\mathbf{Q}}(e_i \otimes 1) = \sum_{j=1}^{r} a_{ij} e_j \otimes 1.$$

Hence ${}_{\mathbf{Z}}f_n^* \otimes \mathrm{id}_{\mathbf{Q}}$ is given by a matrix of integers. Now

$$\Lambda(f) = \sum_n (-1)^n \operatorname{trace} {}_{\mathbf{Q}}f_n^* = \sum_n (-1)^n \operatorname{trace} {}_{\mathbf{Z}}f_n^* \otimes \mathrm{id}_{\mathbf{Q}}.$$

From the Lemma we have

$$\operatorname{trace}({}_{\mathbf{Z}}f_n^* \otimes \mathrm{id}_{\mathbf{Q}}) \equiv \operatorname{trace}({}_{\mathbf{Z}}f_n^* \otimes \mathrm{id}_{\mathbf{Q}})^p \bmod p.$$

Finally,

$$\begin{aligned}\operatorname{trace}({}_{\mathbf{Z}}f_n^* \otimes \mathrm{id}_{\mathbf{Q}})^p &= \operatorname{trace}(J^{-1} \circ {}_{\mathbf{Q}}f_n^* \circ J)^p \\ &= \operatorname{trace}(J^{-1} \circ ({}_{\mathbf{Q}}f_n^*)^p \circ J) = \operatorname{trace}({}_{\mathbf{Q}}f_n^*)^p,\end{aligned}$$

which proves the proposition. □

It can be easily seen that the same proof works in the case of multivalued mappings in the sense of Powers [7].

Our interest in the proposition arose from the attempt of proving the (mod p)-theorem of Zabreĭko and Krasnosel'skiĭ [10] and Steinlein [9] by purely topological means and from its usefulness in asymptotic fixed point theory (cf. Fenske and Peitgen [2] and Peitgen [6]).

We close with a few observations and remarks.

(1) The following is a generalization of results due to Floyd [3], [4] and Halpern [5]. The proofs in [3] and [4] use the Smith special homology groups, and the proof in [5] is based on the Kelley-Spanier characteristic.

THEOREM. *Let X be a compact Lefschetz space such that $\check{H}^*(X; \mathbf{Z})$ is finitely generated and let T be a fixed point free periodic transformation of period $s = p^r$, p prime, operating on X. Then p divides the Euler characteristic $\chi(X)$.*

PROOF. From the proposition we have that

$$\chi(X) = \Lambda(T^{p^r}) \equiv \Lambda(T^{p^{r-1}}) \equiv \cdots \equiv \Lambda(T^p) \equiv \Lambda(T) = 0,$$

where all congruences are mod p. □

We add in passing that the same argument yields the divisibility result in the case when T is a fixed point free transformation on X and ${}_{\mathbf{Q}}(T^s)^* = \mathrm{id}$.

COROLLARY. *Let X be a compact Lefschetz space such that $\check{H}^*(X;\mathbf{Z})$ is finitely generated, and let T be a periodic transformation of period $s = p^r$, p prime, operating on X. Assume that $\chi(X) \not\equiv 0 \bmod p$; then T has a fixed point.*

(2) Set $\lambda_p = p^{-1}(\Lambda(g^p) - \Lambda(g))$. The proposition tells us that λ_p is an integer invariant for g. It seems to be of some interest to determine λ_p especially in view of the following observations. Set

$$\chi_p(X) = \sum_n (-1)^n \dim_{\mathbf{Z}_p} \check{H}^n(X;\mathbf{Z}_p),$$

whenever this number is defined. Assume that X is a finite-dimensional compact Hausdorff space such that $\check{H}^*(X;\mathbf{Z})$ is finitely generated. Notice that a standard argument using Theorem 10 in Spanier [8, p. 335], yields $\chi(X) = \chi_p(X)$. Consider the very special case when g is a fixed point free transformation of prime period p operating on X. Then we have that $\lambda_p = \chi_p(X/G)$. Here G is the group generated by g and X/G is the orbit space. This follows from Floyd [3], [4]. Furthermore, combining another result of Floyd [3] and the proposition, we have that

$$\Lambda(g) \equiv \Lambda(g^p) = \chi(X) \equiv \chi_p(X^G) \bmod p$$

when g is not necessarily fixed point free.

We would like to remark that a purely topological approach for a proof of the (mod p)-theorem for the fixed point index is the following.

Let $f: U \to \mathbf{R}^n$ be continuous, where U is open in $\mathbf{R}^n$. Let p be prime and denote by F (resp. F_p) the fixed point set of f (resp. f^p) in U. Assume that F and F_p are compact and that f^p has no fixed points on the boundary of U and $f(F_p) \subset U$. Then the Krasnosel'skiĭ-Zabreĭko-Steinlein theorem is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd}
H_n(S^n;\mathbf{Z}_p) \arrow[r] \arrow[d] & H_n(S^n, S^n - F;\mathbf{Z}_p) \arrow[r] & H_n(U, U-F;\mathbf{Z}_p) \arrow[r, "(i-f)^*"] & H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\};\mathbf{Z}_p) \arrow[d] \\
H_n(S^n;\mathbf{Z}_p) \arrow[r] & H_n(S^n, S^n - F_p;\mathbf{Z}_p) \arrow[r] & H_n(U, U-F_p;\mathbf{Z}_p) \arrow[r, "(i-f^p)^*"] & H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\};\mathbf{Z}_p)
\end{tikzcd}$$

Using coefficients in $\mathbf{Z}$ instead of $\mathbf{Z}_p$, the rows define a fixed point index for f (resp. f^p) (cf. Dold [1]). It is immediate from Dold's paper that it suffices to prove commutativity of the right rectangle where F is replaced by F_p.

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