ON THE LEFschetz NUMBER FOR ITERATES OF CONTINUOUS MAPPINGS

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Abstract. We give an elementary proof of the (mod p)-congruence between the Lefschetz numbers of a continuous mapping and its pth iterate, p prime, and deduce some results about periodic mappings.

The following proposition can be obtained in a less general situation from theorems of Zabreiko and Krasnosel'skii [10] and Steinlein [9] by using the normalization property of the fixed point index. However, the proofs in [9] and [10] are analytical and quite difficult. Moreover, the normalization property requires some deeper considerations (cf. Dold [1]). In this paper we present a short and appropriate proof using only basic algebraic and topological arguments.

Proposition. Let X be a compact Hausdorff space such that $\tilde{H}^*(X; \mathbb{Z})$ is finitely generated. Then $\Lambda(f) \equiv \Lambda(f^p) \mod p$, whenever $f$ is a continuous map on $X$ and $p$ is prime.

For the proof of the proposition we need the following basic Lemma. Let $A$ be a matrix of integers and let $p$ be prime. Then $\text{trace } A = \text{trace } A^p \mod p$.

Proof. If $A$ is in $\mathbb{Z}^{n \times n}$, we denote by $\Pi : \mathbb{Z}^n \to \mathbb{Z}_p^n$ and by $\pi : \mathbb{Z} \to \mathbb{Z}_p$ the corresponding projections. We have $\Pi(A^p) = (\Pi(A))^p$ and $\pi(\text{trace } A) = \text{trace } \Pi(A)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$. Since the $\lambda_i$ are roots of a polynomial with coefficients in $\mathbb{Z}_p$, then $\sum_{i=1}^n \lambda_i$ is defined in the extension field $\mathbb{Z}_p(\lambda_1, \ldots, \lambda_n)$ of $\mathbb{Z}_p$. Notice that $\mathbb{Z}_p(\lambda_1, \ldots, \lambda_n)$ is a field of characteristic $p$, hence $(\sum_{i=1}^n \lambda_i)^p = \sum_{i=1}^n \lambda_i^p$. Since trace $\Pi(A) = \sum \lambda_i$ and trace $(\Pi(A))^p = \sum \lambda_i^p$ are in $\mathbb{Z}_p$, we have $\sum \lambda_i = (\sum \lambda_i)^p = \sum \lambda_i^p$ in $\mathbb{Z}_p$. Thus we get

$$\pi(\text{trace } A) = \text{trace } \Pi(A) = \text{trace } (\Pi(A))^p = \pi(\text{trace } A^p).$$

Proof of Proposition. From Spanier [8, p. 335], we have that the following diagram with exact rows commutes:

$$\begin{array}{cccc}
0 & \to & \tilde{H}^n(X; \mathbb{Z}) \otimes \mathbb{Q} & \xrightarrow{f^*} & \tilde{H}^n(X; \mathbb{Q}) & \to & \text{Tor}(\tilde{H}^{n+1}(X; \mathbb{Z}), \mathbb{Q}) \\
\downarrow & & \downarrow \text{id}_\mathbb{Q} & & \downarrow & & \\
0 & \to & \tilde{H}^n(X; \mathbb{Z}) \otimes \mathbb{Q} & \xrightarrow{f^*} & \tilde{H}^n(X; \mathbb{Q}) & \to & \text{Tor}(\tilde{H}^{n+1}(X; \mathbb{Z}), \mathbb{Q})
\end{array}$$

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Since Tor(A, Q) = 0 for any abelian group A, we have that J is an isomorphism and, therefore,

\[ \text{trace } (Zf_n^* \otimes \text{id}_Q) = \text{trace } (J^{-1} \circ Qf_n^* \circ J) = \text{trace } (Qf_n^*). \]

Let F be a free subgroup and T the torsion subgroup of \( \tilde{H}^n(X; \mathbb{Z}) \) such that \( \tilde{H}^n(X; \mathbb{Z}) = F \oplus T \), and let \( e_1, \ldots, e_r \) be a basis for F. Then a basis for \( \tilde{H}^n(X; \mathbb{Z}) \otimes \mathbb{Q} \) is given by \( e_i \otimes 1, \ldots, e_r \otimes 1 \). If we choose \( a_{ij} \) in \( \mathbb{Z} \) and \( t \) in T such that \( zf_n^*(e_i) = \sum_{j=1}^{r} a_{ij} e_j + t \), then we have

\[ zf_n^* \otimes \text{id}_Q(e_i \otimes 1) = \sum_{j=1}^{r} a_{ij} e_j \otimes 1. \]

Hence \( zf_n^* \otimes \text{id}_Q \) is given by a matrix of integers. Now

\[ \Lambda(f) = \sum_n (-1)^n \text{trace } Qf_n^* = \sum_n (-1)^n \text{trace } zf_n^* \otimes \text{id}_Q. \]

From the Lemma we have

\[ \text{trace } (zf_n^* \otimes \text{id}_Q) \equiv \text{trace } (zf_n^* \otimes \text{id}_Q)^p \mod p. \]

Finally,

\[ \text{trace } (zf_n^* \otimes \text{id}_Q)^p = \text{trace } (J^{-1} \circ Qf_n^* \circ J)^p \]

\[ = \text{trace } (J^{-1} \circ (Qf_n^*)^p \circ J) = \text{trace } (Qf_n^*)^p, \]

which proves the proposition. \( \square \)

It can be easily seen that the same proof works in the case of multivalued mappings in the sense of Powers [7].

Our interest in the proposition arose from the attempt of proving the (mod p)-theorem of Zabreiko and Krasnosel'skiï [10] and Steinlein [9] by purely topological means and from its usefulness in asymptotic fixed point theory (cf. Fenske and Peitgen [2] and Peitgen [6]).

We close with a few observations and remarks.

(1) The following is a generalization of results due to Floyd [3], [4] and Halpern [5]. The proofs in [3] and [4] use the Smith special homology groups, and the proof in [5] is based on the Kelley-Spanier characteristic.

Theorem. Let X be a compact Lefschetz space such that \( \tilde{H}^*(X; \mathbb{Z}) \) is finitely generated and let T be a fixed point free periodic transformation of period \( s = p^r \), \( p \) prime, operating on X. Then \( p \) divides the Euler characteristic \( \chi(X) \).

Proof. From the proposition we have that

\[ \chi(X) = \Lambda(T^{p^r}) \equiv \Lambda(T^{p^{r-1}}) \equiv \cdots \equiv \Lambda(T^p) \equiv \Lambda(T) = 0, \]

where all congruences are mod \( p \). \( \square \)

We add in passing that the same argument yields the divisibility result in the case when \( T \) is a fixed point free transformation on \( X \) and \( Q(T^s)^* = \text{id} \).
Corollary. Let $X$ be a compact Lefschetz space such that $\tilde{H}^*(X; \mathbb{Z})$ is finitely generated, and let $T$ be a periodic transformation of period $s = p^r, p$ prime, operating on $X$. Assume that $\chi(X) \neq 0 \mod p$; then $T$ has a fixed point.

(2) Set $\lambda_p = p^{-1}(\Lambda(g^p) - \Lambda(g))$. The proposition tells us that $\lambda_p$ is an integer invariant for $g$. It seems to be of some interest to determine $\lambda_p$ especially in view of the following observations. Set

$$
\chi_p(X) = \sum_n (-1)^n \dim_{\mathbb{Z}_p} H^n(X; \mathbb{Z}_p),
$$

whenever this number is defined. Assume that $X$ is a finite-dimensional compact Hausdorff space such that $\tilde{H}^*(X; \mathbb{Z})$ is finitely generated. Notice that a standard argument using Theorem 10 in Spanier [8, p. 335], yields $\chi(X) = \chi_p(X)$. Consider the very special case when $g$ is a fixed point free transformation of prime period $p$ operating on $X$. Then we have that $\lambda_p = \chi_p(X/G)$. Here $G$ is the group generated by $g$ and $X/G$ is the orbit space. This follows from Floyd [3], [4]. Furthermore, combining another result of Floyd [3] and the proposition, we have that

$$
\Lambda(g) = \Lambda(g^p) = \chi(X) = \chi_p(X^G) \mod p
$$

when $g$ is not necessarily fixed point free.

We would like to remark that a purely topological approach for a proof of the (mod $p$)-theorem for the fixed point index is the following.

Let $f : U \to \mathbb{R}^n$ be continuous, where $U$ is open in $\mathbb{R}^n$. Let $p$ be prime and denote by $F$ (resp. $F_p$) the fixed point set of $f$ (resp. $f^p$) in $U$. Assume that $F$ and $F_p$ are compact and that $f^p$ has no fixed points on the boundary of $U$ and $f(F_p) \subset U$. Then the Krasnosel’skii-Zabreiko-Steinlein theorem is equivalent to the commutativity of the following diagram:

$$
\begin{aligned}
H_n(S^n; \mathbb{Z}_p) &\to H_n(S^n, S^n - F; \mathbb{Z}_p) \to H_n(U, U - F; \mathbb{Z}_p) \\
\downarrow &\downarrow (i-f)^* \downarrow (i-f^p)^* \\
H_n(S^n; \mathbb{Z}_p) &\to H_n(S^n, S^n - F_p; \mathbb{Z}_p) \to H_n(U, U - F_p; \mathbb{Z}_p)
\end{aligned}
$$

Using coefficients in $\mathbb{Z}$ instead of $\mathbb{Z}_p$, the rows define a fixed point index for $f$ (resp. $f^p$) (cf. Dold [1]). It is immediate from Dold’s paper that it suffices to prove commutativity of the right rectangle where $F$ is replaced by $F_p$.

References


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