A REMARK ON DERIVATIONS AND
SKEW-DERIVATIONS ON \( \mathcal{D}(M) \)

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Abstract. We give here the corrections of the Proposition of S. Kobayashi and K. Nomizu concerning the derivation and skew-derivation of the algebra of differential forms on a differentiable manifold.

Let \( M \) be an \( n \)-dimensional differentiable manifold, and \( \mathcal{D}(M) \) the space of differential forms of degree \( r \) defined on \( M \). Then with respect to the exterior product, \( \mathcal{D}(M) = \bigoplus_{r=0}^{\infty} \mathcal{D}^r(M) \) forms an algebra over the real field \( \mathbb{R} \). A derivation or a skew-derivation of \( \mathcal{D}(M) \) is a linear mapping of \( \mathcal{D}(M) \) into \( \mathcal{D}(M) \) satisfying the following condition:

1. \( D \): derivation:
   \[
   D (\omega \wedge \omega') = D\omega \wedge \omega' + \omega \wedge D\omega', \quad \text{for } \omega, \omega' \in \mathcal{D}(M).
   \]

2. \( D \): skew-derivation:
   \[
   D (\omega \wedge \omega') = D\omega \wedge \omega' + (-1)^r \omega \wedge D\omega', \quad \text{for } \omega \in \mathcal{D}^r(M), \omega' \in \mathcal{D}(M).
   \]

A derivation or a skew-derivation \( D \) of \( \mathcal{D}(M) \) is said to be of degree \( k \) if it maps \( \mathcal{D}^r(M) \) into \( \mathcal{D}^{r+k}(M) \) for every \( r \). Then the following proposition is given in S. Kobayashi and K. Nomizu [2].

Proposition. (a) If \( D \) and \( D' \) are derivations of degree \( k \) and \( k' \), respectively, then \( DD' - D'D \) is a derivation of degree \( k + k' \).
(b) If \( D \) is a derivation of degree \( k \) and \( D' \) is a skew-derivation of degree \( k' \), then \( DD' - D'D \) is a skew-derivation of degree \( k + k' \).
(c) If \( D \) and \( D' \) are skew-derivations of degree \( k \) and \( k' \), respectively, then \( DD' + D'D \) is a derivation of degree \( k + k' \).
(d) A derivation or a skew-derivation is completely determined by its effect on \( \mathcal{D}^0(M) = \mathcal{F}(M) \) and \( \mathcal{D}^1(M) \).

Recently, through discussions at the Research Institute of Mathematics, Tamkang College, I find that the conclusions (b) and (c) of the Proposition should be corrected as follows.

Proposition. [Case B]. If \( D \) and \( D' \) are derivations of degree \( k \) and \( k' \), then we have:

1. If \( k \) is even, then \( DD' - D'D \) is a skew-derivation of degree \( k + k' \).
2. If \( k \) is odd, then there exists no nonzero derivation nor skew-derivation of the type \( DD' - D'D \) or of the type \( DD' + D'D \).

[Case C]. If \( D \) and \( D' \) are skew-derivations of degree \( k \) and \( k' \), then we have:

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(1) If \( k \) and \( k' \) are both even, then \( DD' - D'D \) is a derivation of degree \( k + k' \).

(2) If \( k \) and \( k' \) are both odd, then \( DD' + D'D \) is a derivation of degree \( k + k' \).

(3) If one of \( k \) or \( k' \) is even and the other is odd, then there exists no nonzero derivation nor skew-derivation of the type \( DD' - D'D \) or of the type \( DD' + D'D \).

The verifications of Cases B and C are straightforward, so we omit them.

N. B. (i) See [1] for a systematic discussion of these questions.


REFERENCES


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