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CONJUGATE POWERS IN *HNN* GROUPS

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ABSTRACT. Our purpose is to show the conjugacy problem is solvable for certain *HNN* groups with many stable letters and in the process investigate conjugate powers in these groups.

Let $G(l,m) = \langle a,b;\ a^{-1}b^l a = b^m\rangle$ where $|l| \neq 1 \neq |m|$, l, $m \neq 0$, and l,m are relatively prime. It is stated in [1, Theorem 1] that if $x \in G(l,m)$ and x^l is conjugate to x^m, then x is conjugate to a power of b. Although the theorem is correct, the proof given in [1] is not. We will prove the following generalization of the above theorem for certain *HNN* groups with many stable letters.

We employ the following concepts in our discussion. A group is a *U-group* if extraction of roots is unique when possible [4b, p. 11]. A subgroup H is *malnormal* in a group K if $gxg^{-1} \in H$, $1 \neq x \in H$ implies that $g \in H$ [3], [6]. A group is said to be *2-free* if every two generator subgroup is free [3].

THEOREM 1. *Let G be an HNN group with base B and stable letters a_i $(i \in I)$ given by*

$$\langle B, a_i;\ \text{rel } B, a_i^{-1} W_i^{p_i} a_i = V_i^{q_i}\ (i \in I)\rangle \tag{I}$$

where W_i, V_i are words in the generators of B, W_i, $V_i \neq 1$ in B and p_i, $q_i \neq 0$ for $i \in I$. Suppose the following conditions are satisfied:

(1) *B is a U-group.*

(2) *W_i, V_i $(i \in I)$ generate malnormal subgroups.*

If $x \in G$ and x^l is conjugate to x^m where $|l| \neq |m|$, then x is conjugate to a power of W_i or a power of V_i for some $i \in I$.

Let us call p_i, q_i $(i \in I)$ the exponents of G. When there are finitely many exponents, $W = W_i = V_i$ for $(i \in I)$ and G satisfies the hypothesis of Theorem 1, we will write $G(p_1, q_1, \ldots, p_k, q_k, W)$. We say that G has *unmeshed* exponents when they are distinct and relatively prime in pairs. Let B be a free product of two finitely generated free groups with infinite cyclic amalgamated subgroups given by

$$\langle b_1, \ldots, b_n, c_1, \ldots, c_m;\ R(b_1, \ldots, b_n) = S(c_1, \ldots, c_m)\rangle. \tag{II}$$

We extend the results of [2] by proving

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THEOREM 2. *Suppose* $G(p_1, q_1, \ldots, p_k, q_k, W)$ *satisfies the following*:

(1) B *is given by* (II) *and is both residually free and 2-free.*

(2) *The exponents of the group are unmeshed.*

Then $G(p_1, q_1, \ldots, p_k, q_k, W)$ *has solvable conjugacy problem.*

Let A consist of those groups B given in (II) whose factors are isomorphic. Further assume that F is an isomorphism from $\langle b_1, \ldots, b_n; \rangle$ to $\langle c_1, \ldots, c_n; \rangle$, R generates its own centralizer in $\langle b_1, \ldots, b_n; \rangle$ and $S = F(R)$. Gilbert Baumslag has shown that the groups of A are residually free and 2-free [4a].

COROLLARY 1. *Suppose* $G(p_1, q_1, \ldots, p_k, q_k, W)$ *satisfies the following*:

(1) B *is in* A.

(2) *The exponents of the group are unmeshed.*

Then $G(p_1, q_1, \ldots, p_k, q_k, W)$ *has solvable conjugacy problem.*

COROLLARY 2. *Suppose* $G(p_1, q_1, \ldots, p_k, q_k, W)$ *satisfies the following*:

(1) B *is the fundamental group of a closed Riemann surface of genus* $g \geqslant 2$ [9, p.91].

(2) *The exponents of the group are unmeshed.*

Then $G(p_1, q_1, \ldots, p_k, q_k, W)$ *has solvable conjugacy problem.*

Let F be the free product of free groups $F_1 * F_2$ with amalgamated subgroups $F_1 \cap F_2$. According to Lemma 2 of [1], if x^l is conjugate to x^m in F where $|l| \neq |m|$, then x is conjugate to some y in the amalgamated subgroup. Prompted by a question from R. Hirshon, the author produced a counterexample to that assertion.

Let $F_1 = \langle b, r_1, r_2; \rangle$ and $F_2 = \langle c, s_1, s_2; \rangle$ be free groups on the indicated generators. Let F be the free product with amalgamation of F_1 and F_2 given by

$$\langle b, r_1, r_2, c, s_1, s_2;\ r_1^{-1} b^2 r_1 = s_1^{-1} c s_1,\ r_2^{-1} b^4 r_2 = s_2^{-1} c s_2 \rangle.$$

We observe that b^2 is conjugate to b^4 in F. By Solitar's Theorem [10, Theorem 4.6], since $b \in R_1$, if b were conjugate in F to some $y \in F_1 \cap F_2$, then b would be conjugate in F_1 to one such $y \in F_1 \cap F_2$. Let N be the normal closure of $F_1 \cap F_2$ in F_1 and note that F_1/N is given by $\langle b, r_1, r_2;\ b^2 = 1 \rangle$ so that $b \notin N$. Hence, b is not conjugate in F_1, and thus in F, to an element of $F_1 \cap F_2$.

PROOF OF THEOREM 1. We let P denote the set of stable letters of G. We may assume without loss of generality that x is P-cyclically reduced [11, p. 21].

If x contains a stable letter, then no cyclic permutation of x^l can be P-parallel to x^m [11, p.19] and so, by Collins Lemma [11, p. 21], x^l is not conjugate to x^m in G. Hence, x lies in B.

Since x^l is conjugate to x^m in G we may choose a reduced word y of the form

$$(*) \qquad y = c_0 a_{i_1}^{e_1} c_1 a_{i_2}^{e_2} \cdots a_{i_n}^{e_n} c_n,$$

$e_i \neq 0$, c_i a word in the generators of B (c_0,c_n possibly empty) such that $y^{-1} x^l y x^{-m} = 1$ in G. It follows from Britton's Lemma [11, p. 14] and our choice of y that $c_0^{-1} x^l c_0 = z^t$ where z is one of W_{i_1}, V_{i_1}. Hence, z^{-t} commutes with $c_0^{-1} x^l c_0$. Now B is a U-group (also called an $\bar{R}$-group), so by the remarks [7, p. 244] $c_0^{-1} x c_0$ commutes with z. Now z is malnormal [3], [6] in B so that z

generates its own centralizer. Hence, $c_0^{-1}xc_0$ is a power of either W_{i_1} or V_{i_1}.

Let L be a subgroup of K. By the *generalized word* (*conjugacy*) *problem* for L in K is meant the problem of determining for arbitrary w in K whether or not w is a member of (conjugate to an element of) K [5, p. 358].

LEMMA 1. *Let B be a countable torsion-free group and $1 \neq w$ an element of B such that*:

(1) *B has solvable word problem.*

(2) *The generalized conjugacy problem for $\langle w \rangle$ in B is solvable.*

Then if $\langle w \rangle$ is a malnormal subgroup of B, the generalized word problem for $\langle w \rangle$ in B is solvable.

PROOF. Since $\langle w \rangle$ is a malnormal subgroup of B, we have for $s \neq 0$, if $x^{-1}w^s x = w^t$, then x is in w and, hence, $s = t$. We need only determine whether $1 \neq w'$ is an element of $\langle w \rangle$. By (2) we may decide whether w' is conjugate to an element of $\langle w \rangle$. Since B is countable, we may enumerate all conjugates of elements of $\langle w \rangle$, say $u_1, u_2, \ldots$. For each u_i we decide whether or not it is equal to w'. In this manner we produce a w^s conjugate to w'. If w' is a member of $\langle w \rangle$, $w' = w^t$. By the preceding remarks it follows that $w' = w^s$.

As a partial generalization of Lemma 9 [11, p. 24], we obtain, from Lemma 1 and Britton's Lemma,

LEMMA 2. *Suppose G is an HNN group given by*

$$\langle B, a_1, \ldots, a_k;\ \text{rel } B,\ a_i^{-1}w_i a_i = v_i,\ i = 1, \ldots, k \rangle$$

where B is a torsion-free group, $w_i, v_i \neq 1$ in B for $i = 1, \ldots, k$ and the following conditions are satisfied:

(1) *B has solvable word problem.*

(2) *The generalized conjugacy problems for $\langle w_i \rangle$ and $\langle v_i \rangle$ in B are solvable where $i = 1, \ldots, k$.*

(3) *$\langle w_i \rangle$, $\langle v_i \rangle$ are malnormal subgroups of B for $i = 1, \ldots, k$.*

Then G has solvable word problem.

Let $p_1, q_1, \ldots, p_k, q_k, l, m$ be nonzero integers. Call m reachable from l with respect to $p_1, q_1, \ldots, p_k, q_k$ if there is a sequence of integers beginning with l and ending with m such that successive terms l_i and l_{i+1} satisfy one of the following conditions:

(1) $l_{i+1} = l_i(q_j/p_j)$ where l_i/p_j is integral.

(2) $l_{i+1} = l_i(p_j/q_j)$ where l_i/q_j is integral.

The reachability problem for $p_1, q_1, \ldots, p_k, q_k$ is to decide for arbitrary integers l,m whether m is reachable from l. The reachability problem for $G(p_1, q_1, \ldots, p_k, q_k, w)$ will be understood to mean the reachability problem for its exponents. In particular, the reachability is solvable for those groups with unmeshed exponents.

LEMMA 3. *Suppose $G(p_1, q_1, \ldots, p_k, q_k, w)$ satisfies the following conditions*:

(1) *B is a finitely presented, residually free and 2-free group.*

(2) *B has solvable conjugacy problem and the generalized conjugacy problem for $\langle w \rangle$ in B is solvable.*

(3) *The reachability problem for the exponents is solvable.*

Then $G(p_1, q_1, \ldots, p_k, q_k, w)$ has solvable conjugacy problem.

PROOF. It follows from Lemma 2, that we may effectively P-reduce and, hence, effectively P-cyclically reduce words in the generators of the group. Without loss of generality we need only consider whether P-cyclically reduced words g and h are conjugate.

Suppose one of g and h contains a stable letter. We may decide whether g is conjugate to h by producing finitely many systems of exponential equations in the manner of [2, p. 268] and determining whether one such system has a solution.

Since B has solvable conjugacy problem we need only consider the case when g and h are words in the generators of B but not conjugate in B.

Suppose $x^{-1}gx = h$ where x is taken to be reduced and of the form $(*)$ given in the proof of Theorem 1. Then it follows from Britton's Lemma that g is conjugate in B to some element w^s in $\langle w^{p_i} \rangle$ or $\langle w^{q_i} \rangle$ and h is conjugate in B to some element w^t in $\langle w^{p_i} \rangle$ or $\langle w^{q_i} \rangle$. Hence, both g and h are conjugate to powers of w and these powers are unique since $\langle w \rangle$ is a malnormal subgroup of B. By Lemma 2 we may produce such w^s and w^t when they exist.

Suppose $y^{-1}w^s y = w^t$ where y is taken to be reduced of the form $(*)$. It follows from Britton's Lemma that there is a sequence $w_1, \ldots, w_{n+1}$ such that $a_{i_j}^{-e_j} c_j^{-1} w_j c_j a_{i_j}^{e_j} = w_{j+1}$ where $j = 1, \ldots, n$ and $w_1 = w^s$, $w_{n+1} = w^t$. If $e_j = 1$, then $c_j^{-1} w_j c_j$ is in $\langle w^{p_r} \rangle$, and if $e = -1$, then $c_j^{-1} w_j c_j$ is in $\langle w^{q_r} \rangle$. Since $\langle w \rangle$ is malnormal in B, we have $c_1^{-1} w^s c_1$ is in $\langle w \rangle$ implying c_1 is in $\langle w \rangle$ and so w_2 is in $\langle w \rangle$. In general, c_j, w_j are in $\langle w \rangle$ for all j. Now choose $z = a_{i_1}^{e_1} \cdots a_{i_n}^{e_n}$ and observe that $z^{-1} w^s z = w^t$. Hence, t must be reachable from s with respect to the exponents of the group.

Therefore, we may determine when words g and h in the generators of B are conjugate in $G(p_1, q_1, \ldots, p_k, q_k, w)$ by producing the appropriate w^s, w^t when they exist and determining whether t is reachable from s.

PROOF OF THEOREM 2. When B is the free product of finitely generated free groups with infinite cyclic amalgamated subgroups, then B has solvable conjugacy problem [8]. It is an immediate consequence of [2, Lemma 3] that the generalized conjugacy problem for $\langle W \rangle$ in B is solvable. Since B is residually free and 2-free and the exponents are unmeshed, it follows from Lemma 3 that $G(p_1, q_1, \ldots, p_k, q_k, W)$ has solvable conjugacy problem.

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