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A GEOMETRIC PROPERTY OF CERTAIN PLANE SETS

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ABSTRACT. Suppose K is a compact subset of the plane of the form $\overline{\Delta(0,1)} \setminus \bigcup_{n=1}^{\infty} \Delta(p_n,r_n)$ where $\overline{\Delta(p_n,r_n)} \subseteq \Delta(0,1)$ for each n and $\overline{\Delta(p_i,r_i)} \cap \overline{\Delta(p_j,r_j)} = \varnothing$ for $i \neq j$. Let $\alpha = \sup_{i \geqslant 1}((r_i + 1)/r_i)$ and define the sets $\partial_* K \equiv \partial\Delta(0,1) \cup [\bigcup_{n=1}^{\infty} \partial\Delta(p_n,r_n)]$ and $F(K) \equiv \{z \in K \setminus \partial_* K\colon z$ is not a point of density of $K\}$. It is proved that if $\alpha < 1$, then $\mathcal{H}^1[F(K)] = 0$, where $\mathcal{H}^1$ denotes Hausdorff one-dimensional measure.

If p is a point in the complex plane $\mathbf{C}$, we denote the set $\{z \in \mathbf{C}\colon |z - p| < r\}$ by $\Delta(p,r)$. The Lebesgue area and Hausdorff one-dimensional measure of a measurable plane set X will be denoted by $\mathcal{L}^2(X)$ and $\mathcal{H}^1(X)$, respectively. The notation $X \triangle Y$ will mean $(X \setminus Y) \cup (Y \setminus X)$.

The term "circular set" will mean a compact subset of $\mathbf{C}$ of the form $\overline{\Delta(0,1)} \setminus \bigcup_{n=1}^{\infty} \Delta(p_n,r_n)$, where $\overline{\Delta(p_n,r_n)} \subseteq \Delta(0,1)$ for all n and $\overline{\Delta(p_i,r_i)} \cap \overline{\Delta(p_j,r_j)} = \varnothing$ for $i \neq j$. This includes a variety of sets considered in the studies of rational approximation and analytic capacity [6], [8].

For each circular set

$$K = \overline{\Delta(0,1)} \setminus \bigcup_{n=1}^{\infty} \Delta(p_n,r_n)$$

with $r_{i+1} \leqslant r_i$ for $i \geqslant 1$, define the *type* of K to be the number $\alpha = \sup_{i \geqslant 1}((r_{i+1})/r_i)$. It is clear that $0 < \alpha \leqslant 1$. Furthermore, the set

$$\partial_* K \equiv \partial\Delta(0,1) \cup \left[\bigcup_{n=1}^{\infty} \partial\Delta(p_n,r_n)\right]$$

(where ∂ denotes topological boundary) is called the *outer boundary* of K. We will also use the notation $K_0 \equiv K \setminus \partial_* K$.

If X is any measurable subset of $\mathbf{C}$, a point $p \in \mathbf{C}$ will be called a *point of density* (respectively, *rarefaction*) of X if

$$\lim_{r \to 0} \frac{\mathcal{L}^2[\Delta(p,r) \cap X]}{\pi r^2} = 1 \quad \text{(respectively, 0).}$$

The purpose of this paper is to prove the following

THEOREM. *Let K be a circular set of type $\alpha < 1$, and let $F(K) = \{z \in K_0\colon z$ is not a point of density of $K\}$. Then $\mathcal{H}^1[F(K)] = 0$.*

The proof requires some preliminaries.

If μ is any compactly supported, complex, regular Borel measure on $\mathbf{C}$, the

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Newtonian potential of μ is defined by

$$U_{|\mu|}(z) = \int \frac{d|\mu|(\zeta)}{|\zeta - z|} .$$

It is well known that $U_{|\mu|}$ is finite almost everywhere with respect to area [2]. For each $z \in \mathbf{C}$ such that $U_{|\mu|}(z) < \infty$, the *Cauchy transform* of μ is defined by

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z} .$$

The function $\hat{\mu}$ is thus defined $\mathfrak{L}^2$ almost everywhere, is analytic off the closed support of μ, and vanishes at infinity.

If f is a complex function with first order partial derivatives, we write

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

where $z = x + iy$.

If μ is a measure as indicated above, it is a consequence of Green's theorem that $\partial\hat{\mu}/\partial\bar{z} = -\pi\mu$ in the sense of distributions. A discussion of this and other properties of the function $\hat{\mu}$ may be found in Chapter 3 of [2].

We have $K = \overline{\Delta(0,1)} \setminus \bigcup_{n=1}^{\infty}\Delta(p_n,r_n)$. Let μ be the measure on K which equals $dz/2\pi i$ on $\partial\Delta(0,1)$ and $-dz/2\pi i$ on $\partial\Delta(p_n,r_n)$, $n = 1,2,\cdots$. Then $\hat{\mu} = \chi_K$ $\mathfrak{L}^2$ almost everywhere, where χ_K denotes the characteristic function of K, and $\partial\hat{\mu}/\partial\bar{z} = -\pi\mu$. Since $\hat{\mu}$ is real, it follows that $\partial\hat{\mu}/\partial z$ is also a measure. Thus, grad $\hat{\mu}$ is a vector-valued measure, which means that the set K has finite perimeter in the sense of DeGiorgi [3], [4].

With Federer [5], we say that a unit vector u is an *exterior normal* of K at z if and only if

$$\lim_{r\to 0} \frac{\mathfrak{L}^2[w\colon |w - z| < r, (w - z)\cdot u < 0, w \notin K]}{r^2} = 0$$

and

$$\lim_{r\to 0} \frac{\mathfrak{L}^2[w\colon |w - z| < r, (w - z)\cdot u > 0, w \in K]}{r^2} = 0$$

where $\cdot$ denotes inner product. Such a unit vector u, if it exists, is uniquely determined by K and z, and is denoted by $\nu(K,z)$. In case no such u exists, $\nu(K,z)$ is the null vector. This defines, for each $z \in \mathbf{C}$, a vector $\nu(K,z) = \nu_1(K,z) + i\nu_2(K,z)$.

Let $N(K)$ be the subset of $\mathbf{C}$ at which $\nu(K,z) \neq 0$, i.e., the set at which an exterior normal to K exists. Since K has finite perimeter, Federer's theorem [5] implies that

$$\frac{\partial}{\partial x}\chi_K = \frac{\partial\hat{\mu}}{\partial x} = -2\pi \operatorname{Re}\mu = \nu_1(K,z)\, d\mathcal{H}^1(z)$$

and

$$\frac{\partial}{\partial y}\chi_K = \frac{\partial\hat{\mu}}{\partial y} = -2\pi \operatorname{Im}\mu = \nu_2(K,z)\, d\mathcal{H}^1(z)$$

in the sense of distributions, where μ is the measure on K defined above. Since μ is supported on $\partial_* K$, this implies the following lemma.

LEMMA 1. *The equality* $\mathcal{H}^1[\partial_* K \triangle N(K)] = 0$ *holds.*

Following Vol'pert [7], we define the *essential boundary* of K to be the set of all points in $\mathbf{C}$ which are neither points of density nor points of rarefaction of K. We write $E(K)$ for the essential boundary of K. The following lemma follows from the theorem of Vol'pert in subsection 4 of [7].

LEMMA 2. *The relation* $E(K) \supseteq N(K)$ *holds. Furthermore,*

$$\mathcal{H}^1[E(K) \setminus N(K)] = 0.$$

Now let $s > \sqrt{1/(1-\alpha^2)}$ be fixed but arbitrary. For any $p \in K_0$ and $r > 0$ define

$$L(p,r) = \mathcal{H}^1\left\{\Delta(p,r) \cap \left[\bigcup_{n=1}^{n_k} \partial\Delta(p_n,r_n)\right]\right\}$$

and

$$A(p,r) = \mathcal{L}^2\left\{\Delta(p,r) \cap \left[\bigcup_{n=1}^{n_k} \Delta(p_n,r_n)\right]\right\}$$

where in each case the union on the right is taken over those finitely many $\Delta(p_n,r_n)$ such that $r_n > r/2s$.

LEMMA 3. *There exists a number* $\beta > 0$ *with the following property. For all* $p \in K_0$ *and* $r > 0$ *such that* $L(p,r) \leqslant \beta r$, *we have*

$$A(p,r) < \frac{\pi}{4s^2}\left(s^2 - \frac{1}{1-\alpha^2}\right)r^2.$$

To prove the lemma, consider a fixed $p \in K_0$ and $r > 0$. Define a function $f\colon [0,r) \to [0,\pi r^2)$ by

$$\begin{aligned} f(t) &= f(p,r;t) \\ &= \sup\left\{\mathcal{L}^2\left[\Delta(p,r) \cap \bigcup_{j=1}^{m} \Delta_j\right] : \mathcal{H}^1\left[\Delta(p,r) \cap \bigcup_{j=1}^{m} \partial\Delta_j\right] \leqslant t\right\} \end{aligned}$$

where the supremum is taken over all finite collections of disks $\{\Delta_j\}$ whose closures are pairwise disjoint and do not contain p. It is clear that $f^{-1}(0) = \{0\}$ and that f is nondecreasing and continuous at the origin. We complete the proof of the lemma by letting β be any positive number such that

$$f(\beta r) < \frac{\pi}{4s^2}\left(s^2 - \frac{1}{1-\alpha^2}\right)r^2,$$

and noting that the same β works for all $r < 1$, by homothetic transformation.

Returning to the proof of the main theorem, we choose β as in Lemma 3 and define $S = \{p \in K_0\colon$ for all sufficiently small $r > 0$, $L(p,r) \geqslant \beta r\}$. Since $|\mu|(S) = 0$, it follows from Theorem 3 of [1] that $\mathcal{H}^1(S) = 0$.

Let $T = K_0 \setminus S$ and consider an arbitrary $p \in T$. There exists a sequence $t_k \searrow 0$ such that $L(p,t_K) < \beta t_K$ and hence

$$A(p,t_k) < \frac{\pi}{4s^2}\left(s^2 - \frac{1}{1-\alpha^2}\right)t_k^2 \quad \text{for } k = 1,2,\cdots.$$

Let t_k be fixed and consider those $\Delta(p_n,r_n)$ with $r_n < t_k/2s$ which intersect $\Delta(p,t_k)$. A simple computation, using the fact that K is of type α, shows that $\mathfrak{L}^2[\Delta(p,t_k) \setminus K] < \pi t_k^2/4$. It follows that p is not a point of rarefaction of K.

The proof of the theorem is completed by summarizing the above results. Lemma 1 implies that $\mathcal{H}^1[K_0 \cap N(K)] = 0$ whence $\mathcal{H}^1[K_0 \cap E(K)] = 0$ by Lemma 2. However, K_0 is the disjoint union of S and T, with $\mathcal{H}^1(S) = 0$. Since T contains no points of rarefaction, we have therefore

$$\mathcal{H}^1[F(K)] = \mathcal{H}^1[F(K) \cap T] = \mathcal{H}^1[E(K) \cap T] = 0. \quad \text{Q.E.D.}$$

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