A GEOMETRIC PROPERTY OF CERTAIN PLANE SETS

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Abstract. Suppose $K$ is a compact subset of the plane of the form
$\Delta(0,1) \setminus \bigcup_{n=1}^{\infty} \Delta(p_n, r_n)$ where $\Delta(p_n, r_n) \subseteq \Delta(0,1)$ for each $n$ and
$\Delta(p_i, r_i) \cap \Delta(p_j, r_j) = \emptyset$ for $i \neq j$. Let $a = \sup_i ((r_i + 1)/r_i)$ and define the
sets $\partial_* K = \partial \Delta(0,1) \cup \bigcup_{n=1}^{\infty} \partial \Delta(p_n, r_n)$ and $F(K) = \{z \in K \setminus \partial_* K: z$ is
not a point of density of $K\}$. It is proved that if $a < 1$, then $\mathcal{H}^1[F(K)] = 0$, where $\mathcal{H}^1$
denotes Hausdorff one-dimensional measure.

If $p$ is a point in the complex plane $C$, we denote the set $\{z \in C: |z - p| < r\}$ by $\Delta(p, r)$. The Lebesgue area and Hausdorff one-dimensional
measure of a measurable plane set $X$ will be denoted by $\mathcal{L}^2(X)$ and $\mathcal{H}^1(X)$, respectively. The notation $X \triangle Y$ will mean $(X \setminus Y) \cup (Y \setminus X)$.

The term “circular set” will mean a compact subset of $C$ of the form
$\Delta(0,1) \setminus \bigcup_{n=1}^{\infty} \Delta(p_n, r_n)$, where $\Delta(p_n, r_n) \subseteq \Delta(0,1)$ for all $n$ and
$\Delta(p_i, r_i) \cap \Delta(p_j, r_j) = \emptyset$ for $i \neq j$. This includes a variety of sets considered in
the studies of rational approximation and analytic capacity [6], [8].

For each circular set $K = \Delta(0,1) \setminus \bigcup_{n=1}^{\infty} \Delta(p_n, r_n)$ with $r_{i+1} < r_i$ for $i \geq 1$, define the type of $K$ to be the number $\alpha$
$= \sup_i ((r_i + 1)/r_i)$. It is clear that $0 < \alpha < 1$. Furthermore, the set
$\partial_* K = \partial \Delta(0,1) \cup \bigcup_{n=1}^{\infty} \partial \Delta(p_n, r_n)$
(where $\partial$ denotes topological boundary) is called the outer boundary of $K$. We
will also use the notation $K_0 = K \setminus \partial_* K$.

If $X$ is any measurable subset of $C$, a point $p \in C$ will be called a point of
density (respectively, rarefaction) of $X$ if
$$\lim_{r \to 0} \frac{\mathcal{L}^2[\Delta(p, r) \cap X]}{\pi r^2} = 1$$
(respectively, 0).

The purpose of this paper is to prove the following

Theorem. Let $K$ be a circular set of type $\alpha < 1$, and let $F(K) = \{z \in K_0: z$
is not a point of density of $K\}$. Then $\mathcal{H}^1[F(K)] = 0$.

The proof requires some preliminaries.

If $\mu$ is any compactly supported, complex, regular Borel measure on $C$, the

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Newtonian potential of $\mu$ is defined by

$$U_{|\mu|}(z) = \int \frac{d\mu(\xi)}{|\xi - z|}.$$  

It is well known that $U_{|\mu|}$ is finite almost everywhere with respect to area [2]. For each $z \in \mathbb{C}$ such that $U_{|\mu|}(z) < \infty$, the Cauchy transform of $\mu$ is defined by

$$\hat{\mu}(z) = \int \frac{d\mu(\xi)}{\xi - z}.$$  

The function $\hat{\mu}$ is thus defined $\ell^2$ almost everywhere, is analytic off the closed support of $\mu$, and vanishes at infinity.

If $f$ is a complex function with first order partial derivatives, we write

$$K = \Re(K-iK) = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad K = \Im(K+iK) = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

where $z = x + iy$.

If $\mu$ is a measure as indicated above, it is a consequence of Green's theorem that $\partial \hat{\mu} / \partial \bar{z} = -\pi \mu$ in the sense of distributions. A discussion of this and other properties of the function $\hat{\mu}$ may be found in Chapter 3 of [2].

We have $K = \Delta(0,1) \setminus \bigcup_{n=1}^{\infty} \Delta(p_n, r_n)$. Let $\mu$ be the measure on $K$ which equals $dz/2\pi i$ on $\partial \Delta(0,1)$ and $-dz/2\pi i$ on $\partial \Delta(p_n, r_n)$, $n = 1, 2, \cdots$. Then $\hat{\mu} = \chi_K \ell^2$ almost everywhere, where $\chi_K$ denotes the characteristic function of $K$, and $\partial \hat{\mu} / \partial \bar{z} = -\pi \mu$. Since $\hat{\mu}$ is real, it follows that $\partial \hat{\mu} / \partial z$ is also a measure. Thus, grad $\hat{\mu}$ is a vector-valued measure, which means that the set $K$ has finite perimeter in the sense of DeGiorgi [3], [4].

With Federer [5], we say that a unit vector $u$ is an exterior normal of $K$ at $z$ if and only if

$$\lim_{r \to 0} \frac{\ell^2 \left[ w : |w - z| < r, (w - z) \cdot u < 0, w \not\in K \right]}{r^2} = 0$$

and

$$\lim_{r \to 0} \frac{\ell^2 \left[ w : |w - z| < r, (w - z) \cdot u > 0, w \in K \right]}{r^2} = 0$$

where $\cdot$ denotes inner product. Such a unit vector $u$, if it exists, is uniquely determined by $K$ and $z$, and is denoted by $\nu(K,z)$. In case no such $u$ exists, $\nu(K,z)$ is the null vector. This defines, for each $z \in \mathbb{C}$, a vector $\nu(K,z) = \nu_1(K,z) + i\nu_2(K,z)$.

Let $N(K)$ be the subset of $\mathbb{C}$ at which $\nu(K,z) \neq 0$, i.e., the set at which an exterior normal to $K$ exists. Since $K$ has finite perimeter, Federer's theorem [5] implies that

$$\frac{\partial}{\partial x} \chi_K = \frac{\partial \hat{\mu}}{\partial x} = -2\pi \Re \mu = \nu_1(K,z) d\mathcal{H}^1(z)$$

and

$$\frac{\partial}{\partial y} \chi_K = \frac{\partial \hat{\mu}}{\partial y} = -2\pi \Im \mu = \nu_2(K,z) d\mathcal{H}^1(z)$$
in the sense of distributions, where $\mu$ is the measure on $K$ defined above. Since $\mu$ is supported on $\partial K$, this implies the following lemma.

**Lemma 1.** The equality $\mathcal{H}^1[\partial K \triangle N(K)] = 0$ holds.

Following Vol'pert [7], we define the essential boundary of $K$ to be the set of all points in $\mathbb{C}$ which are neither points of density nor points of rarefaction of $K$. We write $E(K)$ for the essential boundary of $K$. The following lemma follows from the theorem of Vol'pert in subsection 4 of [7].

**Lemma 2.** The relation $E(K) \supseteq N(K)$ holds. Furthermore,

$\mathcal{H}^1[E(K) \setminus N(K)] = 0.$

Now let $s > \sqrt{1/(1 - \alpha^2)}$ be fixed but arbitrary. For any $p \in K_0$ and $r > 0$ define

$$L(p,r) = \mathcal{H}^1\left[ \Delta(p,r) \cap \bigcup_{n=1}^{n_k} \partial \Delta(p_n, r_n) \right]$$

and

$$A(p,r) = \mathcal{H}^2\left[ \Delta(p,r) \cap \bigcup_{n=1}^{n_k} \partial \Delta(p_n, r_n) \right],$$

where in each case the union on the right is taken over those finitely many $\Delta(p_n, r_n)$ such that $r_n > r/2s$.

**Lemma 3.** There exists a number $\beta > 0$ with the following property. For all $p \in K_0$ and $r > 0$ such that $L(p,r) \leq \beta r$, we have

$$A(p,r) \leq \frac{\pi}{4s^2} \left( s^2 - \frac{1}{1 - \alpha^2} \right) r^2.$$

To prove the lemma, consider a fixed $p \in K_0$ and $r > 0$. Define a function $f: [0,r) \rightarrow [0,\pi r^2]$ by

$$f(t) = f(p,r;t) = \sup \left\{ \mathcal{H}^2\left[ \Delta(p,r) \cap \bigcup_{j=1}^{m} \Delta_j \right] : \mathcal{H}^1\left[ \Delta(p,r) \cap \bigcup_{j=1}^{m} \partial \Delta_j \right] \leq t \right\},$$

where the supremum is taken over all finite collections of disks $\{\Delta_j\}$ whose closures are pairwise disjoint and do not contain $p$. It is clear that $f^{-1}(0) = \{0\}$ and that $f$ is nondecreasing and continuous at the origin. We complete the proof of the lemma by letting $\beta$ be any positive number such that

$$f(\beta r) \leq \frac{\pi}{4s^2} \left( s^2 - \frac{1}{1 - \alpha^2} \right) r^2,$$

and noting that the same $\beta$ works for all $r < 1$, by homothetic transformation.

Returning to the proof of the main theorem, we choose $\beta$ as in Lemma 3 and define $S = \{ p \in K_0 : \text{for all sufficiently small } r > 0, L(p,r) \geq \beta r \}$. Since $|\mu|(S) = 0$, it follows from Theorem 3 of [1] that $\mathcal{H}^1(S) = 0.$
Let $T = K_0 \setminus S$ and consider an arbitrary $p \in T$. There exists a sequence $t_k \searrow 0$ such that $L(p, t_k) < \beta t_k$ and hence

$$A(p, t_k) < \frac{\pi}{4s^2} \left( s^2 - \frac{1}{1 - \alpha^2} \right) t_k^2 \quad \text{for } k = 1, 2, \ldots.$$ 

Let $t_k$ be fixed and consider those $\Delta(p, r_n)$ with $r_n < t_k/2s$ which intersect $A(p, t_k)$. A simple computation, using the fact that $K$ is of type $\alpha$, shows that $\mathcal{E}^2[\Delta(p, t_k) \setminus K] < \pi t_k^2/4$. It follows that $p$ is not a point of rarefaction of $K$.

The proof of the theorem is completed by summarizing the above results. Lemma 1 implies that $\mathcal{H}^1[K_0 \cap N(K)] = 0$ whence $\mathcal{H}^1[K_0 \cap E(K)] = 0$ by Lemma 2. However, $K_0$ is the disjoint union of $S$ and $T$, with $\mathcal{H}^1(S) = 0$. Since $T$ contains no points of rarefaction, we have therefore

$$\mathcal{H}^1[F(K)] = \mathcal{H}^1[F(K) \cap T] = \mathcal{H}^1[E(K) \cap T] = 0. \quad \text{Q.E.D.}$$

**References**

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