A NOTE ON COMMUTATORS AND SINGULAR INTEGRALS

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Abstract. A new approach to the analysis of a certain commutator equation is presented.

1. Introduction. In this note we apply a technique due to Douglas [2] to analyze the solutions to the operator equation

\[ HX - XH = iR^2 \]

where \( H \) is a (possibly unbounded) selfadjoint operator on a (complex) Hilbert space \( \mathcal{H} \), \( R \) is a bounded nonnegative operator on \( \mathcal{H} \), and where the unknown \( X \) is a bounded selfadjoint operator on \( \mathcal{H} \). Our analysis will provide a new and elementary proof of a well-known theorem which is due, in various forms, to Xa Dao-xeng [8], Pincus [5], and Kato [4]. Our approach has certain points of contact with Kato’s but differs from his in that instead of making a detailed analysis of the resolvent of \( H \), we use a basic integral which appears in perturbation theory, and which he used too, to prove a theorem of Putnam [6] by exhibiting an explicit unitary equivalence between \( H \) and multiplication by \( x \) on a direct integral based on Lebesgue measure on \( \mathbb{R} \). This unitary equivalence is then easily seen to be one also between \( X \) and an explicit singular integral operator on the direct integral.

Since \( H \) may be unbounded, what is meant by a solution to equation (1) is open to interpretation. Therefore we specify now that by a solution to equation (1) we shall mean a bounded selfadjoint operator \( X \) such that \( X \text{Dom}(H) \subseteq \text{Dom}(H) \) and such that \((HX - XH)f = iR^2f\) for all \( f \in \text{Dom}(H)\).

2. The solution. We shall write \( U_t = e^{itH}, \ t \in \mathbb{R} \).

Recall that \( f \) belongs to \( \text{Dom}(H) \) precisely when

\[ \frac{d}{dt} U_t f \overset{\text{def}}{=} \lim_{h \to 0} \frac{U_{t+h} f - U_t f}{h} \]

exists, and that for such \( f \), \((d/dt)U_t f = iHU_t f = iU_t Hf\). Consequently, if \( T \) is any bounded linear transformation on \( \mathcal{H} \) which maps \( \text{Dom}(H) \) into \( \text{Dom}(H) \),

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then

\[ \frac{d}{dt} U_t T U_t^* f \overset{\text{def}}{=} \lim_{h \to 0} \frac{U_{t+h} T U_{t+h}^* f - U_t T U_t^* f}{h} \]

exists for all \( f \in \text{Dom}(H) \) and, of course, is \( U_t(i(HT - TH))U_t^* \). Thus a bounded selfadjoint operator \( X \) satisfies equation (1) if and only if \( X \) commutes with \( \{t_i\}_{i \in \mathbb{R}} \) and (\( d/dt \))\( U_t X U_t^* f = -U_t R^2 U_t^* f \) for all \( f \in \text{Dom}(H) \).

**Lemma 2.1.** There is at most one solution \( Q \) to equation (1) which satisfies the equation

\[ \lim_{t \to \infty} U_t Q U_t^* = 0 \]

in the strong operator topology. Moreover, if a solution to equation (1) exists, then a solution \( Q \) satisfying equation (2) also exists, is nonnegative, and every solution \( X \) to equation (1) may be written as \( X = Q + A \) where \( A \) commutes with \( H \).

**Proof.** Suppose \( Q_1 \) and \( Q_2 \) are solutions to equation (1) satisfying equation (2). Then \( Q_1 - Q_2 \) commutes with \( \{U_t\}_{t \in \mathbb{R}} \) and by equation (2) must be zero. Let \( X \) be a solution to equation (1). Then since (\( d/dt \))\( U_t X U_t^* f = -U_t R^2 U_t^* f \), \( f \in \text{Dom}(H) \), \( \{U_t X U_t^*\}_{t \in \mathbb{R}} \) is a decreasing family of bounded selfadjoint operators. Since this family is clearly bounded below, \( \lim_{t \to \infty} U_t X U_t^* \) exists in the strong operator topology and defines a bounded selfadjoint operator \( A \) which clearly commutes with \( \{U_t\}_{t \in \mathbb{R}} \). Hence \( A \) commutes with \( H \) and so \( X - A \) is a nonnegative solution to equation (1) which satisfies equation (2).

The proof is complete.

**Definition 2.2.** If equation (1) has a solution, then the unique solution satisfying equation (2) is called the **principal solution** of equation (1).

We let \( \mathfrak{A} \) denote the closure of the range of \( R \) and we let \( L^2_{\mathfrak{A}}(\mathbb{R}) \) denote the collection of all weakly measurable \( \mathfrak{A} \)-valued functions on \( \mathbb{R} \) which are norm square integrable with respect to Lebesgue measure on \( \mathbb{R} \). The subspace of \( L^2_{\mathfrak{A}}(\mathbb{R}) \) consisting of those functions which vanish a.e. on \((-\infty,0)\) will be denoted by \( L^2_{\mathfrak{A}}([0,\infty)) \) and the projection from \( L^2_{\mathfrak{A}}(\mathbb{R}) \) onto \( L^2_{\mathfrak{A}}([0,\infty)) \) will be denoted by \( P \). Finally, \( \{S_t\}_{t \geq 0} \) denote the unitary group defined on \( L^2_{\mathfrak{A}}(\mathbb{R}) \) by the formula \( (S_t f)(x) = f(x - t), f \in L^2_{\mathfrak{A}}(\mathbb{R}), t \in \mathbb{R} \), and \( \{S_t\}_{t \geq 0} \) denote the semigroup of isometries defined on \( L^2_{\mathfrak{A}}([0,\infty)) \) by restricting \( S_t \), \( t \geq 0 \), to \( L^2_{\mathfrak{A}}([0,\infty)) \); i.e., \( S_t = S_t|L^2_{\mathfrak{A}}([0,\infty)) \), \( t \geq 0 \), where the vertical bar denotes restriction.

**Theorem 1.** Suppose equation (1) has a solution and let \( Q \) be its principal one. Then there is a bounded linear transformation \( C \) from \( \mathfrak{A} \) to \( L^2_{\mathfrak{A}}(\mathbb{R}) \) such that \( C U_t^* = S_t^* C \) for all \( t \in \mathbb{R} \) and such that \( Q = C^* P C \).

**Proof.** First note that from the discussion preceding Lemma 2.1, we may infer that for \( f \in \text{Dom}(H) \),

\[ \int_0^\infty U_t R^2 U_t^* f \, dt = \lim_{T \to \infty} \int_0^T \frac{d}{dt} U_t Q U_t^* f \, dt \]

\[ = Qf - \lim_{t \to \infty} U_t Q U_t^* f = Qf. \]

3 This means that \( A \) commutes with \( H \) if and only if \( A \) commutes with \( \{U_t\}_{t \in \mathbb{R}} \).
Hence, since $\text{Dom}(H)$ is dense and $Q$ is bounded, $\int_0^\infty U_t R^2 U_t^* \, dt$ converges as an improper integral in the strong operator topology to $Q$. For $f \in \mathcal{H}$, we define the $\mathbb{C}$-valued function $\tilde{C}f$ on $[0, \infty)$ by the formula $(\tilde{C}f)(t) = RU_t^* f$. This function is in $L_2^2([0, \infty))$ because

$$\int_0^\infty \| (\tilde{C}f)(t) \|^2 \, dt = \int_0^\infty (U_t^* R U_t R^2 U_t^* f, f) \, dt = (Qf, f) = \| Q^{1/2} f \|^2,$$

and so $\tilde{C}$ is a bounded linear transformation from $\mathcal{H}$ to $L_2^2([0, \infty))$ satisfying the equation $Q = \tilde{C}^* \tilde{C}$. Clearly $\tilde{C}U_t^* = S_t^* \tilde{C}$ for all $t \geq 0$. Since the minimal unitary extension of the semigroup of isometries $\{S_t\}_{t \geq 0}$ is $\{S_t^*\}_{t \in \mathbb{R}}$ and since, $\{U_t\}_{t \geq 0}$ is a semigroup of unitary operators, a well-known theorem (see [3]) implies that there is a unique bounded linear transformation $C$ mapping $\mathcal{H}$ to $L_2^2(\mathbb{R})$ such that $C^*$ extends $\tilde{C}^*$ and such that $CU_t^* = S_t^* C$ for all $t \in \mathbb{R}$. Hence $\tilde{C} = PC$ and $Q = C^* PC$.

3. Applications. In this section we use Theorem I to obtain the results of Xa Dao-xeng, Pincus, and Kato promised in the introduction. Along the way we present a new proof of a theorem of Putnam.

We continue to use the notation established in §2; and if $A$ is a bounded linear transformation (possibly between different Hilbert spaces), we will write $\mathfrak{H}(A)$ for the initial space of $A$; i.e., $\mathfrak{H}(A) = (\ker (A))^\perp$.

**Lemma 3.1.** Suppose equation (1) has a solution and let $Q$ be its principal one. Then $\mathfrak{H}(Q)$ is the smallest subspace of $\mathfrak{H}$ containing the range of $R^2$ and invariant under $\{U_t\}_{t \geq 0}$.

**Proof.** Theorem I could be used for part of the proof, but it is just as easy to proceed without it. Since $U_t Q U_t^* \leq Q$, $t \geq 0$, it follows that ker($Q$) is invariant under $\{U_t^*\}_{t \geq 0}$. Since $R^2 f = \lim_{t \to 0^+} \frac{U_t Q U_t^* f - Qf}{t}$ for all $f$ such that the limit exists, it follows that ker($Q$) $\subseteq$ ker($R^2$). Upon taking orthogonal complements, we find that the range of $R^2$ $\subseteq$ (ker($R^2$))$^\perp$ $\subseteq$ $\mathfrak{H}(Q)$ and that $\mathfrak{H}(Q)$ is invariant under $\{U_t\}_{t \geq 0}$. Now suppose $\mathfrak{M}$ is any subspace containing the range of $R^2$ and invariant under $\{U_t\}_{t \geq 0}$. To show that $\mathfrak{H}(Q) \subseteq \mathfrak{M}$, it suffices to show that $\mathfrak{M}^\perp \subseteq \ker(Q)$. But since $\mathfrak{M}^\perp$ is invariant under $\{U_t^*\}_{t \geq 0}$ and since the range of $R^2$ is contained in $\mathfrak{M}$, $\mathfrak{M}^\perp$ $\subseteq$ ker($R^2$) and $R^2 U_t^* f = 0$ for all $f \in \mathfrak{M}^\perp$ and $t \geq 0$. So, by equation (3), $Qf = \int_0^\infty U_t R U_t^* f \, dt = 0$ for all $f \in \mathfrak{M}^\perp$, and the proof is complete.

The next theorem was first proved by Putnam in [6, Theorem 5]. There he assumed that $H$ is bounded, but later, in [7, §2.13], he proved more general results for unbounded $H$. These are slightly more general than ours, but his method of proof is considerably different from ours (cf. [2, p. 27]).

Recall that the absolutely continuous spectral subspace $\mathcal{H}_{AC}$ for $H$, or $\{U_t\}_{t \in \mathbb{R}}$, is the set of all vectors $f$ in $\mathcal{H}$ such that $d\|E(\lambda)f\|^2$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ where $E$ is the spectral measure for $H$. We say $H$, or $\{U_t\}_{t \in \mathbb{R}}$, is absolutely continuous if $\mathcal{H}_{AC} = \mathcal{H}$.
Theorem II. If equation (1) has a solution, then $\mathcal{H}_{AC}$ contains the range of $R^2$, and so, if the smallest reducing subspace for $\{U_t\}_{t \in \mathbb{R}}$ containing the range of $R^2$ is all of $\mathcal{H}$, then $H$ is absolutely continuous.

Proof. We prove more than we need, but the excess will be used later. Let $Q$ be the principal solution to equation (1) and write $Q = C^* PC$ where $C$ is the bounded linear transformation from $\mathcal{H}$ to $L^2_0(\mathbb{R})$ constructed in the proof of Theorem I. Since $\mathfrak{S}(Q) = \mathfrak{S}(PC)$ is the smallest invariant subspace for $\{U_t\}_{t \geq 0}$ containing the range of $R^2$ by Lemma 3.1, it is clear that since $CU_t^* = S_t^* C$ for all $t$, $\mathfrak{S}(C)$ is the smallest reducing subspace for $\{U_t\}_{t \in \mathbb{R}}$ containing the range of $R^2$. Call this subspace $\mathfrak{M}$, and let $C = YW$ be the polar decomposition of $C$; i.e., $Y = (CC^*)^{1/2}$ and $W$ is the unique partial isometry with initial space $\mathfrak{M}$ such that $C = YW$. Then since $S_t^* C = CU_t^*$ for all $t$, we find that $Y$ commutes with $\{S_t\}_{t \in \mathbb{R}}$ and that $S_t^* W = WU_t^*$ for all $t$. Hence $W$ effects a unitary equivalence between $\{U_t|\mathfrak{M}\}_{t \in \mathbb{R}}$ and the restriction of $\{S_t\}_{t \in \mathbb{R}}$ to a reducing subspace $\mathfrak{M}$ in $L^2_0(\mathbb{R})$. Let $\mathfrak{M}$ denote the Fourier transform regarded as a unitary operator on $L^2_0(\mathbb{R})$; for $t$ in $\mathbb{R}$, let $x_t$ denote the multiplication operator on $L^2_0(\mathbb{R})$ determined by $e^{ix}$; let $\mathfrak{M} = \mathfrak{M}_0$; and let $V = \mathfrak{M} W$. Then $\mathfrak{M} S_t = x_t \mathfrak{M}$, $\mathfrak{M}$ reduces $\{x_t\}_{t \in \mathbb{R}}$, and $V$ is a Hilbert space isomorphism from $\mathfrak{M}$ onto $\mathfrak{M}$ satisfying $V(U_t|\mathfrak{M})V^* = x_t|\mathfrak{M}$ for all $t$. Since $L^2_0(\mathbb{R})$ is a direct integral and since $\mathfrak{M}$ reduces $\{x_t\}_{t \in \mathbb{R}}$ (which generates the algebra of diagonalizable operators on $L^2_0(\mathbb{R})$), we may write $\mathfrak{M}$ as a direct integral $\mathfrak{M} = \int_\mathbb{R} \mathfrak{M}(x) dx$ where $dx$ is Lebesgue measure on $\mathbb{R}$ and where $\mathfrak{M}(x)$ is a subspace of $\mathfrak{M}$ for each $x$ in $\mathbb{R}$ (see [1, Chapter II]). Thus, since for each Borel set $M \subseteq \mathbb{R}$, $V(E(M)|\mathfrak{M})V^* = I_M|\mathfrak{M}$, where $I_M$ denotes the characteristic function of $M$, it follows that $\{U_t|\mathfrak{M}\}_{t \in \mathbb{R}}$ is absolutely continuous. This completes the proof.

We note that the closed support of the direct integral representation of $\mathfrak{M}$, which is the closed support of the spectral measure for $H$, is precisely the spectrum of $H|\mathfrak{M}$, $\Lambda(H|\mathfrak{M})$; i.e., $\mathfrak{M} = \int_\Lambda(H|\mathfrak{M}) \mathfrak{M}(x) dx$.

Theorem III. Let $X$ be a solution to equation (1) and suppose that the smallest reducing subspace for $H$ containing the range of $R^2$ is all of $\mathcal{H}$. Let $V$ be the Hilbert space isomorphism from $\mathcal{H}$ onto $\mathfrak{M} = \int_\Lambda(H) \mathfrak{M}(x) dx$ constructed in the proof of Theorem II so that $V U_t V^* = x_t|\mathfrak{M}$ for all $t$. Then there are selfadjoint decomposable operators on $\mathfrak{M}$ given by functions $M(\cdot)$ and $K(\cdot)$ with $K(x) \geq 0$ a.e. such that

\begin{equation}
\left( V X V^* \right) f(x) = M(x) f(x) - \frac{i}{2\pi} p.v. \int_{\Lambda(H)} K(x) K(y) (x - y)^{-1} f(y) dy \quad a.e.
\end{equation}

for all $f \in \mathfrak{M}$ where $p.v.$ denotes principal value.

Proof. We continue with the notation established in the proof of Theorem II. By Lemma 2.1, the selfadjoint operator $A = X - Q$ commutes with $\{U_t\}_{t \in \mathbb{R}}$ and so $VAV^*$ commutes with $\{x_t|\mathfrak{M}\}_{t \in \mathbb{R}}$. Hence $VAV^*$ is decomposable and therefore is given by a selfadjoint operator-valued function $L(\cdot)$ on $\Lambda(H)$. On the other hand, $VQV^* = (\mathfrak{M} W)(W^* Y) P(YW)(W^* Y) |\mathfrak{M} = (\mathfrak{M} Y \mathfrak{M}^*)(\mathfrak{M} P \mathfrak{M}^*)(\mathfrak{M} Y \mathfrak{M}^*) |\mathfrak{M}$. But $Y$ commutes with $\{S_t\}_{t \in \mathbb{R}}$, so $\mathfrak{M} Y \mathfrak{M}^*$ is
decomposable, and is therefore given by an operator-valued function $K(\cdot)$ on $\Lambda(H)$. Moreover, since $Y \geq 0$, $K(x) \geq 0$ a.e. on $\Lambda(H)$. Finally, observe that $\overline{\delta} P \overline{\delta}^* = (I - i\delta)/2$ where $\delta$ is the Hilbert transform. Putting these things together, and writing $M(\cdot) = L(\cdot) + K^2(\cdot)/2$, we arrive at equation (4). This completes the proof.

REFERENCES


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