PRODUCTS OF SEQUENTIAL SPACES

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Dedicated to Professor Kiiti Morita
on the occasion of his 60th birthday

ABSTRACT. S. P. Franklin introduced the notion of a sequential space and characterized such spaces as being precisely the quotient images of metric spaces.

In this paper we investigate a necessary and sufficient condition for the product of a first countable space with a sequential space to be sequential, and we consider the property "sequential space" in $X^\epsilon$.

1. Introduction. Throughout this paper, by a space we shall mean a regular, $T_1$-space.

The symbol $N$ will refer to the set of natural numbers.

Let us recall that a space $X$ is sequential [2], if a subset $F$ of $X$ is closed whenever $F \cap C$ is closed in $C$ for every convergent sequence $C$ together with its limit point.

Metric spaces, or more generally Fréchet (= Fréchet-Urysohn) spaces are sequential. Sequential spaces are $k$-spaces.

As is well known [2], the product of a sequential space with a separable metric space need not be sequential.

As for the product of a sequential space with a first countable space, our main theorem, which will be established in §3, reads as follows:

THEOREM 1.1. Let $X$ be a Fréchet space, or a sequential space each of whose points is a $G_\delta$-set (or equivalently, a $k$-space each of whose points is a $G_\delta$-set). Let $Y$ be first countable. Then $X \times Y$ is sequential if and only if $X$ is strongly Fréchet, or $Y$ is locally countably compact.

In the necessity, the property "each point of $X$ is a $G_\delta$-set" is essential.

According to Siwiec [12], a space $X$ is called strongly Fréchet (= countably bisequential in the sense of E. Michael [6]) if, whenever $\{F_n; n \in N\}$ (or simply $\{F_n\}$) is a decreasing sequence accumulating at $x$ in $X$, there exist $x_n \in F_n$ such that the sequence $\{x_n; n \in N\}$ (or simply $\{x_n\}$) converges to $x$.

Metric spaces are strongly Fréchet. Strongly Fréchet spaces are Fréchet.

As a special class of sequential spaces, we shall consider symmetrizable spaces.

According to A. V. Arhangel’skii [1], a space $X$ is symmetrizable, if there is a real valued, nonnegative function $d$ defined on $X \times X$ satisfying the
following: (1) $d(x,y) = 0$ iff ($=$ if and only if) $x = y$, (2) $d(x,y) = d(y,x)$, and (3) $A \subseteq X$ is closed iff $d(x,A) \geq 0$ for any $x \in X - A$.

If we replace (3) by (3'): $x \in \bar{A}$ iff $d(x,A) = 0$, then such a space $X$ is called semimetrizable [4].

Metric spaces, or more generally semimetrizable spaces, are symmetrizable. It has been shown [14] that the product of a countable, symmetrizable space with a separable metric space need not be symmetrizable.

In the following theorem, we establish a necessary and sufficient condition for the product of a symmetrizable space with a semimetrizable space to be symmetrizable.

**Theorem 1.2.** Let a symmetrizable space $X$ be paracompact, or more generally meta-Lindelöf (i.e. every open covering has a point-countable open refinement), or have each point a $G_\delta$-set. Let $Y$ be semimetrizable. Then $X \times Y$ is symmetrizable if and only if $X$ is semimetrizable, or $Y$ is locally compact.

As for the property "sequential space" in the product $X^\omega$ of countably many copies of $X$, in §4, we will have

**Theorem 1.3.** Let $X$ have one of the three properties listed below:

(i) $K_{\omega}$-space in the sense of E. Michael [5],

(ii) closed image of a metric space,

(iii) CW-complex in the sense of Whitehead.

If $X^\omega$ is sequential, then $X$ is metrizable.

It follows from [14] that there is a sequential space (in fact, a symmetrizable space) $X$ such that $X^2$ is not sequential.

In this respect, it will be shown that the higher power $X^\omega$ can also behave unpredictably.

That is, there is a space $X$ such that $X^n$ is sequential (in fact, symmetrizable) for all $n \in N$, but $X^\omega$ is not even sequential.

2. Preliminaries. As a weaker condition than "$X$ is strongly Fréchet", we shall often make use of the following condition (C) on $X$.

(C) Let $\{x_n\}$ be a decreasing sequence accumulating at $x \in X$. Then there exist $x_n \in F_n$ such that the sequence $\{x_n\}$ converges to some point $x' \in X$.

**Lemma 2.1.** (A) Let $X$ be a Fréchet space, or a space each of whose points is a $G_\delta$-set. If $X$ satisfies condition (C), then $X$ is strongly Fréchet.

(B) Let $X$ be a symmetrizable space satisfying condition (C). If $X$ is also meta-Lindelöf, or each point of $X$ is a $G_\delta$-set. Then $X$ is semimetrizable.

**Proof.** (A) In case $X$ is Fréchet, in view of the proof of [11, Theorem 5.1], $X$ is strongly Fréchet.

In case each point of $X$ is a $G_\delta$-set, it may be proved directly that $X$ is strongly Fréchet.

(B) We shall prove that $X$ is Fréchet. Because of part (A), we need only consider the case where $X$ is meta-Lindelöf. Let $D$ be a countable subset of $X$. Then the meta-Lindelöf space $\bar{D}$ is separable, hence is Lindelöf. Since $\bar{D}$ is symmetrizable, by [10, Theorem 2] $\bar{D}$ is hereditarily Lindelöf, and hence each point of $\bar{D}$ is a $G_\delta$-set in $\bar{D}$. Since $\bar{D}$ satisfies condition (C), by part (A), $\bar{D}$ is strongly Fréchet. Then $D$ is Fréchet. Thus each countable subset of $X$ is
Fréchet. Hence \( X \) is Fréchet by [6, Proposition 8.7]. Thus \( X \) is first countable, for Fréchet, symmetrizable spaces are first countable [1]. Since first countable, symmetrizable spaces are semimetrizable, \( X \) is semimetrizable.

**Lemma 2.2.** Let \( X \) be sequential. If \( X \) does not satisfy condition (C), then there is a countable, metric space \( Y_0 \) such that \( X \times Y_0 \) is not sequential.

**Proof.** Since \( X \) does not satisfy condition (C), there is a point \( x_0 \) of \( X \), and a decreasing sequence \( \{A_n\} \) accumulating at \( x_0 \) satisfying

(K) If \( x_n \in A_n \), then the sequence \( \{x_n\} \) has no limits. Since \( X \) is sequential and \( x_0 \in A_n \) for \( n \in \mathbb{N} \), by [6, Lemma 8.3 and Proposition 8.5], there is a sequence \( \{C_n\} \) of countable subsets of \( X \) such that \( C_n \subset A_n \) and \( x_0 \in \overline{C_n} \).

Let \( Y_0 = \bigcup_{n=1}^{\infty} (C_n \times \{n\}) \cup \{x_0\} \), and topologize \( Y_0 \) as follows:

Let each point of \( \bigcup_{n=1}^{\infty} C_n \times \{n\} \) be open, and \( \{V_n(x_0)\} \) be a countable local base at \( x_0 \), where \( V_n(x_0) = \bigcup_{i \leq n} (C_i \times \{i\}) \cup \{x_0\} \). Then \( Y_0 \) is a metric space, which is not locally compact. Let \( A = \{(x, (x, n)) \in X \times Y_0 ; n \in \mathbb{N}, x \in C_n\} \).

Then \( (x_0, x_0) \in A - A \). Thus \( A \) is not closed in \( X \times Y_0 \).

Suppose that \( X \times Y_0 \) is sequential. Then a subset \( F \) of \( X \times Y_0 \) is closed whenever \( F \cap (C \times K) \) is closed in \( C \times K \) for every convergent sequence \( C \) in \( X \) and every convergent sequence \( K \) in \( Y_0 \). Let \( C, K \) be convergent sequences in \( X, Y_0 \) respectively, and let \( B = A \cap (C \times K) \). To see that \( B \) is a closed subset of \( C \times K \), let \( z \in C \times K - B \). We need only consider the case \( z = (x, x_0) \). The condition (K) implies that there is \( A_{i_0} \) which contains no elements of \( C \). Then there is a neighborhood \( X \times V_{i_0}(x_0) \) of \( z \) which is disjoint from the set \( B \). Thus \( B \) is closed in \( C \times K \). Hence \( A \) is closed in \( X \times Y_0 \), which is a contradiction. Therefore \( X \times Y_0 \) is not sequential.

**Lemma 2.3.** Let \( X \) be first countable. If \( X \) is not locally countably compact, then the space \( Y_0 \) in Lemma 2.2 is a closed subset of \( X \).

**Proof.** By the hypotheses for \( X \), there is a point \( x_0 \) of \( X \), and a countable local base \( \{U_n\} \) at \( x_0 \) such that each \( U_n \) is not countably compact.

By induction, we can obtain a sequence \( \{C_{n_k}\} \) of countably infinite, discrete subsets of \( X \) such that \( C_{n_k} \subset \overline{U}_{n_k} \), \( C_{n_j} \cap C_{n_k} = \emptyset \) if \( j \neq k \), and \( C_{n_k} \ni x_0 \).

Let \( Z = \bigcup_{k=1}^{\infty} C_{n_k} \cup \{x_0\} \). Then \( Z \) is a closed subset of \( X \) and is homeomorphic to the space \( Y_0 \).

K. Morita [9, Theorem 9.2] has shown that if \( X \times Y \) is a Fréchet space (or equivalently, a hereditarily sequential space [3]), then \( X \) is strongly Fréchet, or \( Y \) is discrete.

As for sequential spaces, from Lemmas 2.2 and 2.3, we have

**Proposition 2.4.** Let \( X \) be sequential, and \( Y \) first countable. If \( X \times Y \) is sequential, then \( X \) satisfies condition (C), or \( Y \) is locally countably compact.

3. Proofs of Theorems 1.1 and 1.2, and some examples.

**Proof of Theorem 1.1.** The necessity follows from Lemma 2.1(A) and Proposition 2.4.

The sufficiency follows from [6, Proposition 4.D.4] and [13, Corollary 2.4].

**Proof of Theorem 1.2.** The sufficiency follows from [14, Corollary 4.4]. So we shall prove the necessity.
For this purpose, let $X \times Y$ be symmetrizable. Then $X \times Y$ is sequential, for symmetrizable spaces are sequential [1].

Suppose $Y$ is not locally compact. Then $Y$ is not locally countably compact, for countably compact, semimetrizable spaces are compact [10, Corollary 2]. Thus $X$ satisfies condition (C) by Proposition 2.4. That $X$ is semimetrizable follows from Lemma 2.1(B).

Now, by the following Remark 3.1, we see that in the necessity of the condition of Theorem 1.1, the assumptions “each point of $X$ is a $G_{\delta}$-set” and “$Y$ is first countable” are essential.

The symbols $R$, $Q$, and $Z$ will denote, respectively, the reals, the rationals, and the integers, all with their usual topologies.

Remark. 3.1. (A) Let $X$ be a compact, sequential space which is not Fréchet. In fact, such a space exists by [3, Example 7.1]. Then $X \times Q$ is sequential by [13, Corollary 2.4]. But $X$ is not strongly Fréchet, nor is $Q$ locally countably compact.

(B). Let $X$ be the quotient space $R/Z$ with $Z$ identified to a point. Then $X$ is a countable CW-complex. Let $Y$ be the countable, symmetrizable space in [14, Example 3.2]. Then a subset $F$ of $Y$ is closed whenever $F \cap C_i$ is closed for every convergent sequence $C_i$ ($i = 0, 1, 2, \ldots$) in $Y$, where $C_0 = \{0\} \cup \{1/n; n \in N\}$, $C_i = \{1/i + 1/n; n \in N\}$. Thus, in view of the proof of [7, Lemma 2.1], $X \times Y$ is sequential. But the Fréchet space $X$ is not strongly Fréchet, nor is $Y$ locally countably compact.

4. The property “sequential space” in $X^\omega$.

Proposition 4.1. Let $X^\omega$ be sequential. Then $X$ satisfies condition (C).

Proof. In case $X$ is countably compact, it is easy to check that a sequential space $X$ satisfies condition (C).

In case $X$ is not countably compact, the space $N$ may be regarded as a closed subset of $X$. Since $X \times N^\omega$ is a closed subset of $X^\omega$, $X \times N^\omega$ is sequential. Hence $X$ satisfies condition (C) by Proposition 2.4.

By Lemma 2.1(A) and Proposition 4.1, we have

Proposition 4.2. Let $X$ be Fréchet, or each point of $X$ be a $G_{\delta}$-set. If $X^\omega$ is sequential, then $X$ is strongly Fréchet.

Lemma 4.3. Let $X$ have one of the properties (i), (ii) and (iii) in Theorem 1.3. If $X$ is strongly Fréchet, then $X$ is metrizable.

Proof. From [6, Theorem 9.11 and Corollary 9.10], we need only prove case (iii).

Let $\mathcal{D} = \{e_\alpha; \alpha \in A\}$ be the collection of cells in $X$. We shall prove the collection $\mathcal{D}' = \{e_\alpha; \alpha \in A\}$ is point-finite.

Suppose that there is a point $x_0 \in X$ such that infinitely many elements of $\mathcal{D}$ contain the point $x_0$. So we may assume each $e_\alpha$ ($\alpha \in A$) contains the point $x_0$. Let $F_n = \bigcup_{\alpha \in \Lambda} e_\alpha - \bigcup_{\beta \in \Gamma} e_\beta$. Then $x_0 \in F_n$ for $n \in N$. Since $X$ is strongly Fréchet, there exist $x_n \in F_n$ such that a sequence $\{x_n\}$ converges to some point $x' \in X$. Let us put $K = \{x_n; n \in N\} \cup \{x'\}$. Then $K$ meets infinitely many elements of $\mathcal{D}$. While, $K$ is a compact subset of a CW-complex
Thus $K$ meets only a finite number of elements of $\mathcal{D}$. This is a contradiction. Hence the collection $\mathcal{S}$ is point-finite.

By a similar method, we can see that for each point $x$ of $X$, assuming only these $\bar{e}_{a_i} (i = 1, 2, \ldots, l)$ contain the point $x$, there is a neighborhood $U$ of $x$ such that $U \subset \bar{e}_{a_1} \cup \cdots \cup \bar{e}_{a_l}$. Since $\bar{e}_{a_1} \cup \cdots \cup \bar{e}_{a_l}$ is metric, $U$ is metric. Hence $X$ is locally metrizable. Since a CW-complex is paracompact [9], $X$ is metrizable.

**Proof of Theorem 1.3.** From the hypothesis for $X$, each point of $X$ is a $G_\delta$-set. Thus, by Lemma 2.1(A), Proposition 4.1 and Lemma 4.3, $X$ is a metric space.

From the following example, we see that the product $X^\omega$ of a sequential (symmetrizable) space $X$ need not be sequential (symmetrizable) even if, for all $n \in N$, $X^n$ is sequential (symmetrizable).

**Example 4.4.** Let $X$ be the symmetrizable space $Y$ in Remark 3.1(B). Then, in view of the proof of [7, Lemma 2.1], for all $n \in N$, $X^n$ is sequential and hence is symmetrizable by [14, Theorem 4.2], while $X$ is an $K_\sigma$-space but is not metrizable. Then, by Theorem 1.3, $X^\omega$ is not even sequential and hence is not symmetrizable.

**REFERENCES**


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