PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 54, January 1976

PROJECTIVE EXTENSIONS OF BANACH ALGEBRAS

ANDY R. MAGID[1]

ABSTRACT. It is shown that if A is a commutative Banach algebra and B a faithful A-algebra finitely generated and projective as an A-module then B can be endowed with a structure of Banach algebra extending that of A.

In [**AH**, Theorem 3.6, p. 205], R. Arens and K. Hoffman established that if A is a commutative (real or complex) Banach algebra and if f in $A[X]$ is a monic polynomial then there exists a norm on the A-algebra $B = A[X]/(f)$ that extends the norm on A and under which B becomes a Banach algebra itself.

In [**M**, Theorem 4, p. 138], this result was extended to show that a result similar to the above holds whenever B is any commutative A-algebra which is finitely generated and projective as an A-module. (The proof in [**M**] relies on the result of [**AH**].)

The purpose of the present note is to provide a brief, elementary proof of a similar theorem where B is not required to be commutative. (Of course, this result has those of [**AH**] and [**M**] as corollaries.)

We will use the following conventions throughout: all rings and algebras have identities. If R is a ring and M an R-module then $\mathrm{End}_R(M)$ denotes the ring of R-linear endomorphisms of M. If R is a ring and S a subset of R then R^S denotes the commutant of S in R, i.e., $R^S = \{x \in R: xs = sx$ for all $s \in S\}$. If B is a Banach space then $L(B)$ denotes the Banach algebra of bounded linear operators from B to itself.

The proof of the theorem will use the following facts from commutative algebra.

LEMMA 1. *Let R be a commutative ring and T a faithful R-algebra which is finitely generated and projective as an R-module. Then*:

(i) *There is a finitely generated projective faithful R-module P such that $T \otimes_R P$ is a free R-module* [**B**, *Theorem* 4.6, *p.* 476].

(ii) *With notation as before,* $\mathrm{End}_R(T) \otimes_R \mathrm{End}_R(P)$ *is isomorphic to* $\mathrm{End}_R(T \otimes_R P)$ [**DI**, *Corollary* 2.6, *p.* 15] *and this latter is a matrix ring by* (i).

(iii) *With notation as before, regard* $\mathrm{End}_R(P)$ *as a subring of* $\mathrm{End}_R(T \otimes_R P)$ *via inclusion on the second factor. Then the commutant of* $\mathrm{End}_R(P)$ *in* $\mathrm{End}_R(T \otimes_R P)$ *is isomorphic to* $\mathrm{End}_R(T)$ [**DI**, *Theorem* 4.3, *p.* 57].

(iv) *Let r: $T \to \mathrm{End}_R(T)$ be the left regular representation:$r(t)(x) = tx$. Then the commutant of $r(T)$ in $\mathrm{End}_R(T)$ is $\mathrm{End}_T(T)$, which is canonically isomorphic to T.*

Received by the editors January 20, 1975.
AMS (MOS) subject classifications (1970). Primary 46J05, 13B99.
Key words and phrases. Projective module, Banach space, commutant.
[1] Partially supported by NSF Grant GP-37051.

We also need the following two elementary remarks about Banach algebras.

LEMMA 2. *Let B be a Banach algebra and S a subset of B. Then B^S is a closed subalgebra of B.*

PROOF. For $b \in B$, let R_b, L_b: $B \to B$ be defined by $R_b(x) = bx$ and $L_b(x) = xb$. Both R_b and L_b are bounded linear operators on B.Thus $B^{\{b\}} = \text{kernel}(L_b - R_b)$ is closed in B, and hence $B^S = \bigcap\{B^{\{b\}}: b \in S\}$ is closed in B.

Now suppose A is a commutative Banach algebra. Let $A^{(n)}$ be the free A-module of rank n, and define a norm on $A^{(n)}$ be the free A-module of rank n, and define a norm on $A^{(n)}$ by $\|\langle x_1, \ldots, x_n\rangle\| = \sup\|x_i\|$. This norm makes $A^{(n)}$ into a Banach space.

LEMMA 3. *Let $A, A^{(n)}$ be as above. For each $a \in A$, the map $l(a)$ in $\text{End}_A(A^{(n)})$ defined by $l(a)(x) = ax$ is continuous, and the commutant of $l(A)$ in $L(A^{(n)})$ is $\text{End}_A(A^{(n)})$.*

PROOF. If $x = \langle x_1, \ldots, x_n\rangle$, $\|ax\| = \sup\|ax_i\| \leqslant \sup\|a\| \|x_i\| = \|a\| \|x\|$; thus $l(a)$ is bounded. It is clear that $L(A^{(n)})^{l(A)}$ is contained in $\text{End}_A(A^{(n)})$. For the opposite inclusion, we need that every element h in $\text{End}_A(A^{(n)})$ is bounded [**M**, Lemma 2, p. 138]: it is easy to see that if (a_{ij}) is the matrix of h in the standard basis of $A^{(n)}$ then $\|hx\| \leqslant (n \cdot \sup\|a_{ij}\|)\|x\|$ and, hence, h is bounded.

THEOREM. *Let A be a commutative Banach algebra and B a faithful A-algebra which is finitely generated and projective as an A-module. Then there is a norm on B which extends the norm on A and under which B is complete.*

PROOF. We regard A as a subring of B. We identify B with $\text{End}_B(B)$ in $\text{End}_A(B)$ as Lemma1(iv). Let P be a finitely generated projective R-module such that $B \otimes_A P = A^{(n)}$ as in Lemma 1(i) and identify $\text{End}_A(B)$ with a subring of $\text{End}_A(A^{(n)})$ as in Lemma 1(iii). Finally, make $A^{(n)}$ a Banach space and regard $\text{End}_A(A^{(n)})$ as a subring of $L(A^{(n)})$ as in Lemma 3. Then $\text{End}_A(A^{(n)})$ is the commutant of $l(A)$ in $L(A^{(n)})$ by Lemma 3, $\text{End}_A(B)$ is the commutant of $\text{End}_A(P)$ in $\text{End}_A(A^{(n)})$ by Lemma 1(iii), and B is the commutant of $r(B)$ in $\text{End}_A(B)$ by Lemma 1(iv). Now $L(A^{(n)})$ is a Banach algebra and by repeated applications of Lemma 2 we see that the subalgebra B of $L(A^{(n)})$ is closed and, hence, complete.

We have already seen in Lemma 3 that if a is in A, $\|l(a)\| \leqslant \|a\|$. If $y = \langle 1,0, \ldots, 0\rangle$ in $A^{(n)}$, $\|y\| = 1$ and $\|l(a)y\| = \|ay\| = \|a\|$. Thus $\|l(a)\| \geqslant \|a\|$, and hence $\|l(a)\| = \|a\|$. It follows that the norm on B extends that on A.

We remark that the part of the proof of the theorem which relies on Lemma 1(i) is unnecessary if the algebra B in the hypothesis of the theorem is free as an A-module. This is the case, for example, when $B = A[X]/f$ where f is a monic polynomial over A, which is the situation considered in [**AH**].

Also, the theorem easily implies a similar result for the case where A is just a commutative normed ring. We state this as a

COROLLARY. *Let A be a commutative normed ring and B a faithful A-algebra which is finitely generated and projective as an A-module. Then there is a norm on B extending the norm on A.*

PROOF. Let $\hat{A}$ be the completion of A, and let $\hat{B} = \hat{A} \otimes_A B$. Since B is A-flat we have an injection $B \to \hat{B}$ compatible with the injection $A \to \hat{A}$. Since $\hat{B}$ is a faithful $\hat{A}$-algebra which is finitely generated and projective as an $\hat{A}$-module, by the theorem the norm on $\hat{A}$ extends to $\hat{B}$. The restriction of this norm on $\hat{B}$ to B is the desired extension.

REFERENCES

[AH] R. Arens and K. Hoffman, *Algebraic extension of normed algebras*, Proc. Amer. Math. Soc. **7**(1956), 203–210. MR**17**, 1113.

[B] H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968. MR**40** #2736.

[DI] F. R. DeMeyer and E. Ingraham, *Separable algebras over commutative rings*, Lecture Notes in Math., vol. 181, Springer-Verlag, Berlin and New York, 1971. MR**43** #6199.

[M] A. Magid, *Algebraically separable extensions of Banach algebras*, Michigan Math. J. **21**(1974), 137–143.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OKLAHOMA, NORMAN, OKLAHOMA 73069

Current address: Department of Mathematics, University of Illinois, Urbana, Illinois 61801