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A UNIQUENESS RESULT FOR TOPOLOGICAL GROUPS

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ABSTRACT. We give a rapid proof of a general result which has an easy corollary that the p-adic integers have a unique topology in which they are a complete separable metric group.

In [1] Corwin showed that the p-adic integers have a unique topology in which they are a nondiscrete locally compact group. The purpose of this note is to give a rapid proof of the following general theorem. It contains most of Corwin's result as a special case. The proof is by methods different than those employed by Corwin.

THEOREM 1. *Let G be a complete separable Abelian metric group. For each integer n, let $n \cdot G = [na|a \text{ in } G]$. Suppose that the translates of the $n \cdot G$ generate the Borel structure of G. Then G has a unique topology in which it is a complete separable metric group.*

PROOF. It is not a priori obvious that the $n \cdot G$ are Borel subsets of G. However, let L and K be complete separable metric groups, and $\psi: L \to K$ a continuous homomorphism. Then ψ induces a continuous one-to-one homomorphism of $L/\text{kernel } \psi$ onto $\psi(L)$. Since $L/\text{kernel } \psi$ is also a complete separable metric group, Souslin's theorem implies that $\psi(L)$ is a Borel subset of K. In particular, the $n \cdot G$ are Borel subsets of G.

Let G' be a complete separable metric group which is isomorphic to G as an abstract group but perhaps has a different topology. Let $\phi: G' \to G$ be the natural identification. But for each integer n, $n \cdot G' = \phi^{-1}(n \cdot G)$ is a Borel subset of G'. Hence, since the translates of the $n \cdot G$ generate the Borel structure of G, we have that ϕ is a Borel mapping. Hence, by Kuratowski [2, p. 400], there exists a set P of first category in G' such that $\phi|G' - P$ is continuous.

The proof of the theorem may now be completed in standard fashion. We claim that ϕ is actually continuous on all of G'. To show this, let a_n $(n \geq 1)$ and a be elements of G' such that $a_n \to a$ (as $n \uparrow \infty$). Now if Q is the set which is the union of $a^{-1} \cdot P$ and $a_n^{-1} \cdot P$ $(n \geq 1)$, Q is again a set of the first category. Hence, $G' - Q$ is nonempty. Let b be an element of $G' - Q$. Then ab is in $G' - P$ and $a_n b$ is in $G' - P$ $(n \geq 1)$. But $a_n b \to ab$. Hence, $\phi(a_n b) \to \phi(ab)$, and so $\phi(a_n) = \phi(a_n b) \cdot \phi(b^{-1}) \to \phi(ab) \cdot \phi(b^{-1}) = \phi(a)$. Hence, ϕ is a continuous one-to-one mapping of G' onto G. Hence, since both

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G and G' are complete separable metric groups, ϕ is actually a topological isomorphism. Q.E.D.

Note that Corwin's result is an immediate corollary. If G is the p-adic integers, then the $n \cdot G$ are open subgroups of G which form a basis at the identity. Hence, the translates of the $n \cdot G$ generate the topology and thus the Borel structure of G.

Bibliography

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