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EXTREMAL AND MONOGENIC ADDITIVE SET FUNCTIONS

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ABSTRACT. The extreme points of the convex set of all additive set functions on a field, which coincide on a subfield are characterized by a simple approximation property. It is proved that a stronger approximation property is characteristic for a so-called monogenic additive set function on a field, which can be generated uniquely by an additive set function on a subfield. Finally it is shown that a simple decomposition property must hold if the convex set above has a finite number of extreme points.

1. **Definitions.** In the terminology of [4] let $ba(\Sigma, \nu, \Sigma')$, denote the set of all $\mu \in ba(S, \Sigma')$ with $\mu \geq 0$ and $\mu(S) = 1$, such that $\mu|\Sigma = \nu$, where Σ' is a field of subsets of a set S, Σ is a subfield of Σ' and $\nu \in ba(S, \Sigma)$ with $\nu \geq 0$ and $\nu(S) = 1$. The set $ca(\Sigma, \nu, \Sigma')$, where Σ and Σ', $\Sigma \subset \Sigma'$, denote σ-fields and ν is a probability measure on Σ, is defined in the same way.

2. **Main results.** From the techniques of Douglas [3] it follows that $\mu \in ca(\Sigma, \nu, \Sigma')$ is an extreme point iff $L_1(S, \Sigma, \nu)$ is dense in the set $L_1(S, \Sigma', \mu)$ of all (equivalence classes of) μ-integrable functions with respect to the norm topology. The completeness of $L_1(S, \Sigma, \nu)$ implies that $\mu \in ca(\Sigma, \nu, \Sigma')$ is an extreme point iff for all $A \in \Sigma'$ there exists $B \in \Sigma$ with $\mu(A \triangle B) = 0$. Stone's representation theorem yields the following obvious generalization to $ba(\Sigma, \nu, \Sigma')$:

THEOREM 1. *It holds that $\mu \in ba(\Sigma, \nu, \Sigma')$ is an extreme point iff for all $A \in \Sigma'$ and $\epsilon > 0$ there exists $B \in \Sigma$ with $\mu(A \triangle B) < \epsilon$.*

PROOF. Let (S_1, Σ_1') in the terminology of [4] denote the Stonian space of (S, Σ') and τ the isomorphism of Σ' onto the field Σ_1' of open and closed subsets of S_1; Σ_1'' is defined to be the σ-field generated by Σ_1'. Since τ induces an isomorphism T of $ba(S, \Sigma)$ onto $ca(S_1, \Sigma_1')$, $\mu \in ba(\Sigma, \nu, \Sigma')$ is an extreme point iff $T(\mu) \in ca(\Sigma_2'', \nu'', \Sigma_1'')$ is an extreme point, where Σ_2'' is the σ-field which is generated by the field $\Sigma_2' = \tau(\Sigma)$, and $\nu'' \in ca(S_1, \Sigma_2'')$ is the (uniquely determined) extension of ν' defined by $\nu'(B) = \nu(\tau^{-1}(B))$, $B \in \Sigma_2'$. Furthermore $\mu' = T(\mu) \in ca(\Sigma_2'', \nu'', \Sigma_1'')$ is an extreme point iff for all $A_1 \in \Sigma_1''$ there exists $B_1 \in \Sigma_2''$ with $\mu'(A_1 \triangle B_1) = 0$. Finally for all $A_1 \in \Sigma_1''$, resp. $B_1 \in \Sigma_2''$, and $\epsilon > 0$, there is a $C_1 \in \Sigma_1'$, resp. $D_1 \in \Sigma_2''$, such that $\mu'(A_1 \triangle C_1) < \epsilon$, resp. $\mu'(B_1 \triangle D_1) < \epsilon$, holds [1, p. 21]. From this Theorem 1 follows if one notices that τ is an isomorphism of Σ' onto Σ_1' and

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that with the help of the symmetric difference and μ' a pseudometric with respect to Σ_1" is defined.

REMARKS. 1. If one chooses for Σ the trivial field $\{\emptyset, S\}$, one gets a result of Choquet [2, p. 245], who characterized the extreme points of the set of all $\mu \in ba(S, \Sigma)$ with $\mu \geq 0$ and $\mu(S) = 1$ by the $\{0,1\}$-valued ones.

2. If Σ' can be generated by adjoining a system Γ of subsets of S to Σ, the approximation property of Theorem 1 is satisfied for all $A \in \Sigma$ iff it holds for all $G \in \Gamma$, because the class of all $A \in \Sigma'$ with this approximation property is a field which contains Σ and Γ.

3. If Σ' can be generated by adjoining a countable system Γ of subsets of S to Σ, the set of extreme points of $ba(\Sigma, \nu, \Sigma')$ is a G_δ-set in the weak* topology of $ba(S, \Sigma')$ without Choquet's metrizability assumptions as the example $S = \mathbf{N}$, $\Sigma = \{\emptyset, S\}$ and $\Sigma' = \wp(S)$ shows because of [4, p. 426].

EXAMPLES. 1. Let Γ be equal to $\{G\}$. Then Łós and Marczewski [6] have shown, that μ_i, $i = 1, 2$, defined by $\mu_i(B_1 G + B_2 G^c) = \nu^*(B_1 G) + \nu_*(B_2 G^c)$ for $i = 1$, resp., $\nu_*(B_1 G) + \nu^*(B_2 G^c)$ for $i = 2$ and for all $B_j \in \Sigma, j = 1, 2$, are elements of $ba(\Sigma, \nu, \Sigma')$, where ν^*, resp. ν_*, is the outer, resp. inner, measure of ν. From Theorem 1 and the following remarks one concludes, that they are extreme points of $ba(\Sigma, \nu, \Sigma')$.

2. If S is a compact topological space, Σ, resp. Σ', σ-fields of Baire, resp. Borel, subsets of S, then a Baire (probability) measure ν_0 can be uniquely extended to a regular Borel measure μ_0, from which follows that μ_0 is an extreme point of $ca(\Sigma, \nu_0, \Sigma')$ (it is not difficult to prove that μ_0 is even an exposed point in the sense that there is a set $A_0 \in \Sigma'$, take, for example, the support of μ_0, such that $\mu_0(A_0) = 1 > \mu(A_0)$ for all $\mu \in ca(\Sigma, \nu_0, \Sigma')$ with $\mu \neq \mu_0$). Hence for all $A \in \Sigma'$ there is a $B \in \Sigma$ with $\mu_0(A \triangle B) = 0$. This is a known result (Berberian [1, p. 221]).

Whereas in the case $ca(\Sigma, \nu, \Sigma')$ the set of extreme points may be empty (take, for example, for S the set of real numbers, $\Sigma = \{B \subset S \mid B$, resp. B^c, is countable$\}$, Σ' is defined to be the set of Borel subsets of S, and ν is defined by $\nu(B) = 0$, resp. 1, if B, resp. B^c, is countable), from the theorem of Kreĭn and Milman and from $ba(\Sigma, \nu, \Sigma') \neq \emptyset$ follows

COROLLARY. *Every $\nu \in ba(S, \Sigma)$ with $\nu \geq 0$ and $\nu(S) = 1$ can be extended to $\mu \in ba(S, \Sigma')$ with $\mu \geq 0$ and $\mu(S) = 1$, such that for all $A \in \Sigma$ and $\epsilon > 0$ there is a $B \in \Sigma$ with $\mu(A \triangle B) < \epsilon$.*

REMARK. For the extension μ of ν in the Corollary, it holds that the closures (in the topology of the set of real numbers) of the range of ν, resp. of μ, coincide. The existence of extensions with this property are proved by Sikorski and Tarski (see [6]). If the extension μ of ν is unique, it follows from $ba(\Sigma, \nu, \Sigma') \neq \emptyset$ and Example 1 that a stronger approximation property for μ holds.

THEOREM 2. *$ba(\Sigma, \nu, \Sigma') = \{\mu\}$ holds iff for all $A \in \Sigma'$ and $\epsilon > 0$ there exists $B_i \in \Sigma$, $i = 1, 2$, with $B_1 \subset A \subset B_2$ and $\nu(B_2 \setminus B_1) < \epsilon$.*

REMARKS. 1. Theorem 2 holds in the case $ca(\Sigma, \nu, \Sigma') = \{\mu\}$ if it is possible to extend probability measures on σ-fields Σ'' ($\Sigma \subset \Sigma'' \subset \Sigma'$) to Σ', for example if S is countable (see [5]). But even in the case where S is compact and Σ, resp. Σ', is the set of Baire, resp. Borel, subsets of S, the countably additive version of Theorem 2 is false. Choose, for example, a set S' with card $(S') = \aleph_1$ and equip S with the discrete topology. If $S = S' \cup \{\infty\}$ denotes the one point compactification of S, then Σ consists of all countable subsets of S' and their complements with respect to S and $\Sigma' = \wp(S)$. Now on $\wp(S)$ exist because of a theorem of Ulam [7] only discrete probability measures, which implies that for the Dirac measure δ_∞ on Σ' it holds $ca(\Sigma, \delta_\infty|\Sigma, \Sigma') = \delta_\infty$, but does not have the approximation property of Theorem 2. This answers two questions of Berberian [1, p. 233], whether a monogenic Baire measure is always completely regular in the sense of Theorem 2 and whether Σ is equal to Σ' if all Baire measures are monogenic.

2. Theorem 2 implies that $\nu \in ba(S, \Sigma)$, $\nu \geq 0$, $\nu(S) = 1$ can be uniquely decomposed in the following way: $\nu = a\nu_1 + (1-a)\nu_2$, $a \in [0,1]$, where $\nu_1 \in ba(S,\Sigma)$, $\nu_1 \geq 0$, $\nu_1(S) = 1$ can be uniquely extended to $\mu_1 \in ba(S, \Sigma)$ with $\mu_1 \geq 0$ and with $\nu_2 \in ba(S,\Sigma)$, $\nu_2 \geq 0$, $\nu_2(S) = 1$ such that ν_2 is singular with respect to all $\nu_1' \in ba(S,\Sigma)$ with this property. Furthermore ν_1 is given by: $a\nu_1(B) = \inf\{\nu_*(A_1) + \cdots + \nu_*(A_n) | A_i \in \Sigma$ pairwise disjoint, $i = 1, \ldots, n$, $\bigcup_{i=1}^n A_i = B\}$, where ν_* is the inner measure of ν (restricted to Σ).

Finally a simple decomposition property will be derived, which ν must have if $ba(\Sigma, \nu, \Sigma')$ has at most r extremal points. For this purpose let $B_i \in \Sigma$, $i = 1, \ldots, s$, pairwise disjoint with $\bigcup_{i=1}^s B_i = S$ and $\{\mu_1, \ldots, \mu_s\}$, $s \leq r$, the set of extreme points of $ba(\Sigma, \nu, \Sigma')$. Then $\mu \in ba(\Sigma, \nu, \Sigma')$ defined by $\mu(A) = \mu_1(B_1 A) + \cdots + \mu_s(B_s A)$ for all $A \in \Sigma'$ has the approximation property in Theorem 1 which implies that there is a $t \in \{1, \ldots, s\}$ such that $\mu_t = \mu_i$ on $B_i \Sigma$ for all $i \in \{1, \ldots, s\}$. Since this property holds for the restriction of ν to $B\,\Sigma$ for an arbitrary $B \in \Sigma$ one yields

THEOREM 3. *If $ba(\Sigma, \nu, \Sigma')$ has at most $r \geq 2$ extreme points, then there exist $a_i \in [0,1]$, $i = 1, \ldots, s$, with $\sum_{i=1}^s a_i \leq 1$, and $\{0,1\}$-valued finitely additive set functions ν_i, $i = 1, \ldots, s$, on Σ, such that $\nu = \sum_{i=1}^s a_i \nu_i + (1 - \sum_{i=1}^s a_i)\nu_0$ holds, where $s = \binom{r}{2}$ and $\nu_0 \in ba(S, \Sigma)$, $\nu_0 \geq 0$, $\nu_0(S) = 1$, and ν_0 has the approximation property in Theorem 2.*

PROOF. Induction with respect to r together with a maximal (with respect to inclusion) system $\{B_i | B_i \in \Sigma$, $i \in I$, pairwise disjoint and for all $i \in I$ the corresponding B_i can be decomposed in at least r pairwise disjoint subsets with positive measure $\nu\}$ implies Theorem 3.

REMARK. By simple examples it is seen that the number $s = \binom{r}{2}$ is minimal in the representation.

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