PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 54, January 1976

# ANY UNITARY PRINCIPAL SERIES REPRESENTATION OF $\mathrm{GL}_n$ OVER A $p$-ADIC FIELD IS IRREDUCIBLE

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ABSTRACT. This paper proves that for the group $\mathrm{GL}_n$ over the $p$-adics every unitary principal series representation is irreducible.

We have an application [3] for the titled fact. Since we were unable to find any proof in the literature, we are presenting two here.

Let $F$ be a nonarchimedean local field and let $G = \mathrm{GL}_n(F)$. Let $D$ be the diagonal subgroup of $G$ and $U$ the upper unipotent matrices. Let $B = D \cdot U$ be the group of all nonsingular upper triangular matrices and write $\delta$ for the modular function of $B$. (If $d_l b$ is a left Haar measure for $B$, then $\delta(b) d_l b$ is a right Haar measure.) Let $K$ be a maximal compact subgroup of $G$ and recall that $G = KB$.

Let $\chi$ be any (unitary) character of $D$ and regard it as a character of $B$. The induced representation $\pi_\chi = \mathrm{Ind}_B^G(\chi\delta^{1/2})$ is called the (unitary) principal series representation attached to $\chi$. To describe the unitary representation $\pi_\chi$ explicitly, we let $\mathcal{H}_\chi$ denote the Hilbert space of all complex-valued measurable functions $h$ on $G$ such that $h(gb) = \chi^{-1}(b)\delta^{-1/2}(b)h(g)$ $(g \in G, b \in B)$ and such that $\int_K |h(k)|^2 dk < \infty$. Then $\pi_\chi$ is just left translation in $\mathcal{H}_\chi$: $(\pi_\chi(x)h)(g) = h(x^{-1}g)$ $(h \in \mathcal{H}_\chi;\ g, x \in G)$.

The Theorem we wish to prove twice is:

THEOREM. *$\pi_\chi$ is irreducible.*

Our first proof is essentially folklore, and depends on results of Mackey.

PROOF. (1). Let $P$ be the subgroup of $G$ which consists of all matrices with zero entries in the top row except in the first place. Then $P \cdot B$ is a dense open subset of $G$ whose complement has Haar measure zero. The Mackey subgroup theorem [4], therefore, implies that $\pi_\chi | P = \mathrm{Ind}_{B\cap P}^P(\chi\delta^{1/2})$. Write $N$ for the unipotent radical of $P$ and note that $(B \cap P) \cdot N = H$ is a closed solvable subgroup of $P$. Inducing in stages, we obtain $\pi_\chi | P = \mathrm{Ind}_H^P(\mathrm{Ind}_{B\cap P}^H(\chi\delta^{1/2}))$.

First, observe that $(\mathrm{Ind}_{B\cap P}^H(\chi\delta^{1/2}))|N$ is the regular representation of $N$. Next, letting $\hat{N}$ denote the Pontrjagin dual of the abelian group $N$ and noting that $N$ is a normal subgroup of $P$, we obtain an action $\mathrm{Ad}^*$ of $P$ on $\hat{N}$. The restricted action $\mathrm{Ad}^* | B \cap P$ has an open dense orbit $\mathcal{U} \subset \hat{N}$ and the measure of $\hat{N} - \mathcal{U}$ is zero. Let $\eta \in \mathcal{U}$ and let $C$ (resp. $Q$) be the isotropy

Received by the editors November 11, 1974.

*AMS (MOS) subject classifications* (1970). Primary 22E50; Secondary 22E35.

*Key words and phrases*. Principal series, unitary representation, irreducibility, $p$-adic group.

[1] The authors would like to thank the Sonderforschungsbereich "Theoretische Mathematik" at the University of Bonn for supporting this research.

subgroup of $\eta$ in $H$(resp. $P$). Of course, $N \subseteq C \subseteq Q$. There is a unique extension of $\eta$ to $C$ which agrees with $\chi\delta^{1/2}$ on $C \cap B$. Call this extension $\phi\delta^{1/2}$. It is easy to see that $\mathrm{Ind}_C^H(\phi\delta^{1/2}) \sim \mathrm{Ind}_{B\cap P}^H(\chi\delta^{1/2})$. Thus, $\pi_\chi|P \sim \mathrm{Ind}_Q^P(\mathrm{Ind}_C^Q(\phi\delta^{1/2}))$, and by Mackey's irreducibility criterion [4], $\pi_\chi|P$ is irreducible if and only if $\mathrm{Ind}_C^Q(\phi\delta^{1/2})$ is irreducible.

To check this irreducibility let $\tilde{Q}$ be a complement to $N$ in $Q$, so that $Q = \tilde{Q} \times N$ (semidirect product). Put $\tilde{C} = C \cap \tilde{Q}$ and $\tilde{\phi} = \phi|\tilde{C}$. Then $\mathrm{Ind}_C^Q(\phi\delta^{1/2})|\tilde{Q} = \mathrm{Ind}_{\tilde{C}}^{\tilde{Q}}(\tilde{\phi}\delta^{1/2})$ and, as one sees, the irreducibility of this last representation is our Theorem for $GL_{n-1}$ instead of $GL_n = G$. To complete the proof apply mathematical induction.

Note that we have, in fact, established that $\pi_\chi|P$ is irreducible. An easy proof for the case $n = 2$ results from appropriately simplifying the above argument.

Our second proof depends upon a deep result of Harish-Chandra and the fact that our Theorem is known to be true for $GL_2$.

PROOF. (2). For $PGL_2$ the Theorem is proved in [6, Theorem 3.4] by a simple computation; the case of $GL_2$ is not different. A proof for $GL_2$ which is more in the spirit of what will follow results from observing that if $\chi$, regarded as a character of the diagonal group $D$, is fixed by the Weyl group, then (for $GL_2$!) the Plancherel measure is zero at $\chi$ [5], [6]. This implies that $\pi_\chi$ is irreducible [2, Corollary 5.4.2.3].

Now let $n \geqslant 3$. The theorem of Harish-Chandra which we shall use is too complicated to explain in detail here, so we give a simplified version. We set $\omega = \chi$ and $P = B$ and obtain a space $L(\chi, B)$ as defined in [1, §11]. We also need the operators ${}^\circ c_{B|B}(s : \chi : 0)$ (cf. ibid.). Let $W(\chi)$ be the subgroup of the Weyl group $W = N_G(D)/Z_G(D)$ such that $\chi^s = \chi$. Harish-Chandra's theorem [2, Theorem 5.5.3.3] implies that: There is a representation

$$s \mapsto {}^\circ C_{B|B}(s : \chi : 0)$$

of $W(\chi)$ on $L(\chi, B)$ which is trivial if and only if $\pi_\chi$ is irreducible.

Observe that $W(\chi)$ is always (in the case of $GL_n$!) generated by reflections corresponding to reduced roots of $G$. Without loss of generality, we may in fact assume that these reduced roots are simple. Thus, it suffices to show that ${}^\circ c_{B|B}(s : \chi : 0)$ is the identity when $s$ is a reflection in $W(\chi)$ with respect to a fixed simple root.

Let $D'$ be the maximal subtorus of $D$ which is fixed by $s$. There is a maximal parabolic subgroup $P' = M'N'$ of $G$, where $M' = Z_G(D')$ is isomorphic to a product of $GL_m$'s ($m < n$), $N'$ is the unipotent radical of $P'$, and $P' \supset B$. The group ${}^*B = M' \cap B$ is a Borel subgroup of $M'$, i.e. it is concretely a product of upper triangular subgroups. The space $L(\chi, {}^*B) \supset L(\chi, B)$. We need the following fact [2, Theorem 5.3.5.3]:

$${}^\circ c_{B|B}(s : \chi : 0) = {}^\circ c_{*B|*B}(s : \chi : 0)|_{L(\chi, B)}.$$

But now the proof is finished, because $\mathrm{Ind}_{*B}^{M'}(\delta_{*B}^{1/2}\chi) = \pi'_\chi$ is a representation of the principal series of $M'$, so it is equivalent to a tensor product of principal series representations of $GL_m$'s ($m < n$); thus $\pi'_\chi$ is irreducible. Therefore, ${}^\circ C_{*B|*B}(s : \chi : 0)$ is already the identity on $L(\chi, {}^*B)$.

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